# Detailed Lectures Content

**Whenever a proof could be asked at the exam it is said: with proof.**

- April 13th. Definition of bounded operator on a Hilbert space (we always assume Hilbert spaces separable). Definition of compact operators on a Hilbert space: definition F.3.3 of [1], or theorem VI.13 of [2]. The compact operators are a double sided ideal: proposition F.3.4 of [1]. Example of a compact operator: proposition 9.3 of [3]. Four theorems on the compact operators (without proofs): the Fredholm Alternative, corollary on pag 203 of [2], the Riesz-Schauder theorem, theorem VI.15 of [2] together with theorem 6 pag 238 of [4], Hilbert-Schmidt, theorem VI.16 of [2], the canonical form of compact operators, theorem VI.17 of [2].

- April 20th. Theorems on trace class and Hilbert-Schmidt operators: theorem VI.18 of [2], definition of trace class operator page 207 of [2], theorem VI.19 of [2], theorem VI.21 of [2], Corollary page 209 of [2], definition of Hilbert-Schmidt operator page 210 of [2], theorem VI.22 of [2], theorem VI.23 of [2].

- April 27th. Introduction to Fredholm operators: theorem 7.1.1 of [1], definition 7.1.2 of [1], recalling the closed graph lemma (lemma F.3.5 of [1]), proposition 7.1.3 of [1] with proof, definition 7.1.4 of [1], theorem 7.1.6 of [1] with proof (first half), lemma 7.2.1 of [1], corollary 7.2.2 of [1] with proof.

- May 4th. More of Fredholm operators: proposition 7.2.3 of [1] with proof, proposition 7.2.4 of [1] with proof, proposition 7.2.6 of [1] with proof.

- May 11th. Proof of the second half of theorem 7.1.6 of [1] (not asked at the exam).

- May 18th. Definition of the Bott index of a pair of unitary matrices and of a pair of invertible matrices: definition 4, lemma 5 (with proof) and section 3.1 of [5]. Homotopy invariance of the Bott index: lemma 6 of [5] with proof.

- June 1st. Definition of an insulating Hamiltonian on a two-dimensional torus: definition 1 of [5]. Definition of the distance among two points on the torus, and of the operators of "position" on the torus, equation 2 of [5].

- June 8th. Lemma 3 of [5] with proof.

- June 15th. Showing how $\|[X,H]\| \le O(R\|H\|)$ follows from $\|[e^{2\pi i \frac{X}{L}}, H]\| \le O\left(\frac{R}{L}\|H\|\right)$. Equation 32 in Lemma 8 of [5] with proof. Theorem 11 of [5] with proof.

- June 22nd. Introduction to the physics of the integer quantum Hall effect (IQHE): mostly section IID of [6]. Introducing the Landau Hamiltonian: section 111 and 112 of [7]. Stressing the role of disorder and the appearance of localized states in the density of states, Fig.3 of [6], that explains the plateaus of the transverse conductance in Fig.2 of [6]. Discussing the different geometrical settings and mentioning the corresponding indices that explain the quantization of the transverse conductance: infinite plane (Fredholm index [6], index of two projections and Fredholm index [8], spectral localizer [9]), finite slab (not known in 2D, mentioning the 1D SSH model [10]), two-dimensional torus (Bott index). Discussing the meaning of an index theorem taking as example the Bott index and the transverse conductance: theorem 11 of [5].

**Second Part: Finite spin systems, introduction to the main models (Ising, $XY$ and $XXZ$), proof of Lieb-Robinson bounds**

- July 6th. Definition of a tensor product of finite dimensional Hilbert spaces and its scalar product, product basis and lemma 1.50 of [11], properties of the tensor product of matrices: theorem 1.55 of [11], definition of partial trace of a matrix acting on a tensor product of spaces, matrix elements of the partial trace (equation (17.9) of [12]) and invariance under conjugation by a local unitary. Algebra of Pauli matrices: definition, commutation and anticommutation properties, evaluation of $\exp(i\alpha\sigma_a)$. Hamiltonian of the transverse field quantum Ising model: qualitative discussion of ferromagnetic, antiferromagnetic and disordered phases.

- July 13th. Explanation of the different concepts on the support of an operator (equation (2) of [13]) in a spin system, showing that operators with disjoint supports commute. Qualitative analysis of the phase diagram (at zero temperature) of the finite-size XXZ model (as in figure 1 of arxiv 2406.11955). Statement of the Lieb-Robinson bound for one-dimensional lattice systems and nearest neighbour Hamiltonians, with proof based on Appendix B of [14].

- July 20th. Completing the proof of the Lieb-Robinson bound based on Appendix B of [14]: equations (B1)-(B17). Showing that the formulation of the L-R bound through the restriction of the support of the Heisenberg evolution of an operator is equivalent to the usual formulation of the bound on the norm of a commutator, and in turn is equivalent to the bound on $\Delta_j(t)$, as in equation (B1) of [14].

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