

Mathematical Statistical Physics

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Sheet 4

Exercise 1: Let ρ_t be a solution of the Vlasov equation. Show that $\|\rho_t\|_p$ is conserved under time evolution for any $p \in [1, \infty]$.

Show that there are solutions of the heat-equation μ_t such that $\|\mu_t\|_p$ is not conserved, but for $p = 1$.

Exercise 2: (a) Let $\rho : \mathbb{R}^6 \rightarrow \mathbb{R}_0^+$ be a probability density with $\|\rho\|_\infty < \infty$. Let $(\bar{x}_j, \bar{v}_j) \in \mathbb{R}^6$ be a sequence of random points in configuration space which are i.i.d. with respect to ρ . Show that for any function $g : \mathbb{R}^6 \rightarrow \mathbb{R}$ with Lipschitz-norm $\|g\|_L < \infty$ it holds that

$$\lim_{N \rightarrow \infty} \mathbb{P} \left(\left| \frac{1}{N} \sum_{j=1}^N g(\bar{x}_j, \bar{v}_j) - \int g(x, v) \rho(x, v) d^3x d^3v \right| > \varepsilon \right) = 0 \quad \forall \varepsilon > 0$$

(If you cite the respective Theorems the solution to this is actually only one line)

(b) Assume now, that $(x_j, v_j) \in \mathbb{R}^6$ is the Newtonian trajectory describing the microscopic system we used to derive the Vlasov equation. For this exercise we drop the time-index, since it is not relevant here. In class we showed that

$$\lim_{N \rightarrow \infty} \mathbb{P} \left(\|(X, V) - (\bar{X}, \bar{V})\|_\infty > \varepsilon \right) = 0 \quad \forall \varepsilon > 0.$$

Use this and part (a) of this exercise to show that

$$\lim_{N \rightarrow \infty} \mathbb{P} \left(\left| \frac{1}{N} \sum_{j=1}^N g(x_j, v_j) - \int g(x, v) \rho(x, v) d^3x d^3v \right| > \varepsilon \right) = 0 \quad \forall \varepsilon > 0$$

Exercise 3: Consider a system of N identical, distinguishable particles. These particles are distributed among discrete energy levels $\epsilon_1, \epsilon_2, \dots, \epsilon_k$, where n_i is the number of particles in the i -th energy level.

The total number of particles N and the total internal energy U are fixed:

$$\sum_{i=1}^k n_i = N, \quad \sum_{i=1}^k n_i \epsilon_i = U$$

Find the macro-state (n_1, n_2, \dots, n_k) with maximal entropy.

What density-distribution do you expect for the density of the atmosphere as a function of the height, what density-distribution do you expect for a gas in a box as function of the kinetic energy of the particles?

Please submit the exercise sheet in pairs or groups of three via URM by 2:00 PM on May 22nd, 2026.