MATHEMATISCH-NATURWISSENSCHAFTLICHE FAKULTÄT
DER UNIVERSITÄT TÜBINGEN SoSe 2026

Mathematical Statistical Physics

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Sheet 6

Exercise 1: Prove, that a measure-preserving transformation T is ergodic if and only if every T-invariant function $f \in L^2(\mu)$ is constant almost everywhere.

Exercise 2: Consider the circle $X = S^1 = \mathbb{R}/\mathbb{Z}$ equipped with the standard Lebesgue measure μ. Let $T_\alpha : X \to X$ be the rotation by $\alpha \in \mathbb{R}$, defined by $T_\alpha(x) = x + \alpha \pmod 1$.

(a) Suppose $\alpha = p/q$ is rational (with p, q coprime). Describe the orbit of any point $x \in X$. What is the exact recurrence time for any point $\omega \in \Omega$?

(b) Suppose α is irrational.

- Prove that the orbit $\{T_\alpha^n(x) : n \in \mathbb{N}\}$ is dense in S^1 for every x.
- Prove ergodicity of T.

Exercise 3: Imagine a frictionless billiard table shaped like a square, $[0, 1] \times [0, 1]$. A tiny ball is shot from the origin at an angle θ such that $\tan(\theta)$ is an irrational number. The ball bounces off the walls elastically (the angle of incidence equals the angle of reflection).

Suppose you place a small square pocket E of area A anywhere on the table. Prove that the long-term fraction of time the ball spends inside E is exactly equal to A, regardless of where E is placed.

Please submit the Exercise sheet in pairs or groups of three via URM by 2:00 PM on June 12^{th}, 2026.