MATHEMATISCH-NATURWISSENSCHAFTLICHE FAKULTÄT
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Mathematical Statistical Physics

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Sheet 7

Exercise 1: In the **canonical ensemble**, a system is in thermal contact with a heat bath at temperature T, but the number of particles N is strictly fixed. The microscopic states are weighted by the Boltzmann factor $e^{-\beta\epsilon}$, where ϵ is the energy of the state and $\beta = \frac{1}{k_B T}$.

In the **grand canonical ensemble**, the system can also exchange particles with a reservoir.

Think, for example, of an ideal gas of fixed volume V, particle number N and energy E in a box at equilibrium. Inside the box there is a smaller volume. The number of particles n in this smaller volume as well as the energy ε these particles have will fluctuate, of course.

Show that in this case the micro-states are weighted by the factor $e^{-\beta(E_i(N)-\mu N)}$ for some $\mu \in \mathbb{R}$.

Exercise 2: Consider a system of N independent, distinguishable, non-interacting spins fixed in space and placed in an external magnetic field B. Each spin i has a magnetic moment μ and can point either parallel or anti-parallel to the field. The energy of an individual spin is given by:

$$\epsilon_i = \begin{cases} -\mu B & \text{(spin up, } \uparrow) \\ +\mu B & \text{(spin down, } \downarrow) \end{cases} \tag{1}$$

The total energy of the system is fixed in a microcanonical shell at exactly $E = -(N_\uparrow - N_\downarrow)\mu B$, where $N_\uparrow$ is the number of up spins and $N_\downarrow$ is the number of down spins ($N = N_\uparrow + N_\downarrow$).

a) Find the total number of microstates $\Omega(N, E)$ available to the system for a given energy E.

b) Determine the microcanonical probability $\mathbb{P}(\{\sigma_1, \ldots, \sigma_N\})$ of finding the system in a specific microstate $\{\sigma_i\}$ that is compatible with the macrostate (N, E).

c) What is the probability $\mathbb{P}_{\text{single}}(\uparrow)$ that a specific, pre-selected single spin (e.g., spin 1) is pointing up? Express your answer in terms of N and $N_\uparrow$.

Exercise 3: A classical, one-dimensional harmonic oscillator of mass m and angular frequency ω is in thermal equilibrium with a heat bath at absolute temperature T. The Hamiltonian of the oscillator is:

$$H(q,p) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 \tag{2}$$

where q is the position and p is the momentum.

a) Write down the continuous probability density function $\rho(q,p)$ in phase space using the canonical Boltzmann factor.

b) Calculate the normalized probability density $d(q)$ of finding the oscillator at a specific position q, regardless of its momentum (i.e., integrate out p).

Please submit the exercise sheet in pairs or groups of three via URM by 2:00 PM on June 19^{th}, 2026.