Mathematical Statistical Physics

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Sheet 9

Exercise 1: Consider a classical system described by a continuous phase space Γ with coordinates (q, p) where the energy at point (q, p) is given by (the Hamiltonian) $H(q, p)$. The system is in thermal equilibrium with a heat reservoir at temperature T, where $\beta = \frac{1}{k_B T}$. The canonical partition function is given by:

$$Z(\beta) = \int_\Gamma e^{-\beta H(q,p)}\, dq\, dp$$

(a) Show that the variance of the energy fluctuations, defined as $\sigma_E^2 = \langle E^2 \rangle - \langle E \rangle^2$, can be expressed directly in terms of the second derivative of the logarithm of the partition function. Specifically, prove that:

$$\sigma_E^2 = \frac{\partial^2 \ln Z}{\partial \beta^2}$$

(b) Using the mathematical result derived in (a), relate the energy variance σ_E^2 to the constant-volume heat capacity, defined as $C_V = \frac{\partial \langle E \rangle}{\partial T}$. Show that:

$$\sigma_E^2 = k_B T^2 C_V \qquad \text{(Fluctuation-Dissipation Relation)}$$

Exercise 2: Consider an ideal gas consisting of N non-interacting monoatomic particles. Use the results of exercise 1) for the following tasks.

(i) Calculate the explicit relative energy fluctuation, defined as the ratio of the standard deviation to the mean energy, $\frac{\sigma_E}{\langle E \rangle}$.

(ii) Analyze the asymptotic behavior of this ratio in the limit $N \to \infty$. What physical conclusion can be drawn from this regarding the equivalence of the microcanonical and canonical ensembles for macroscopic systems?

Please submit the exercise sheet in pairs or groups of three via URM by 2:00 PM on July 3^{rd}, 2026.