Exam on Mathematical Statistical Physics

Peter Pickl
Fachbereich Mathematik, Universität Tübingen

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- Please write your name at the top right of every page you submit. Always start a new task on a new page. No sheet of paper should contain solutions to multiple tasks; using several sheets for a single task is, of course, permitted.
- Allowed aids are writing utensils and one A4 sheet of paper, which may be handwritten on both sides (a legal "cheat sheet"). Paper for the solutions will be provided. A non-programmable calculator is permitted.
- If you have any communication devices with you (mobile phone, laptop, etc.), they must be kept stored away, e.g., in a backpack with a closed zipper. The same applies to any material related to the course (books, lecture notes, etc.).
- Each task carries the same number of points.
- Additional paper can be requested if needed.
- The working time is 100 minutes.

Name, given Name:...

Matriculation Number:...

Degree Course:...

1	2	3	4	5	Σ

Exercise 1: Consider a system of N identical, indistinguishable Fermions. These particles are distributed among discrete energy levels $\epsilon_1, \epsilon_2, \ldots, \epsilon_k$, where n_i is the number of particles in the i-th energy level and g_i the degeneracy of each energy level.

The total number of particles N and the total internal energy U are fixed:

$$\sum_{i=1}^{k} n_i = N, \quad \sum_{i=1}^{k} n_i \epsilon_i = U$$

Find the macro-state $(n_1/g_1, n_2/g_2, \ldots, n_k/g_k)$ with maximal entropy, assuming high degeneracy g_i of the different energy levels.

Exercise 2: Consider two particles on a one dimensional torus $[0, 2\pi[$. The wave function desribing the two particles is given by the symmetrized product of $\phi = C$ and $\eta = Ce^{ikx}$ for some $k \in \mathbb{Z}$ and a normalization-constant C.

Calculate the probability that the distance between the two particles is less than $\frac{\pi}{2}$ as a function of k.

What happens to this probability as k tends to infinity.

Exercise 3: When deriving the validity of the Hartree equation in class, we proved a Grönwyll-type estimate for $\langle \Psi_t, (1 - |\phi_t(x_1)\rangle\langle\phi_t(x_1)) \, \Psi_t\rangle =: \langle \Psi_t, q_t^1 \Psi_t\rangle$. Among other we had to control the expression

$$I := \left|\langle \Psi_t, p_t^1 p_t^2 \left(V \star |\phi_t|^2(x_1) - V(x_1 - x_2)\right) q_t^1 p_t^2 \Psi_t\rangle\right| \; .$$

Show that $I = 0$.

Exercise 4: Let $g_t(x) := C_t e^{-\frac{x}{2\sigma_t^2}}$. Find C_t and σ_t such that g_t solves the heat equation

$$\frac{d}{dt} g_t = \Delta g_t \; .$$

Show that $\rho \star g_t$ also solves the heat equation for any $\rho \in L^1$.

Exercise 5: Consider a system of N particles subject to free Newtonian motion, i.e. for any $t \in \mathbb{R}$ and any $j \in \{1, 2, ..., N\}$ $x_t^j = x_0^j + v^j t$ and $v^j = \text{const}$ for any $t \in \mathbb{R}$ and any $j \in \{1, 2, ..., N\}$.

Assume that initially the macro-state of the system is defined via the phase-space density $\rho_0(x, v)$.

(a) Show that the entropy of the system stays constant

(b) Show that the local temperature of the gas (i.e. the variance of the velocities) tends to zero as $N \to \infty$.