

Mathematical Statistical Physics

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Sheet 11

Exercise 1: Let

$$H = -\Delta + V(x)$$

, for some continuous function $V : \mathbb{R}^3 \to \mathbb{R}$.

(a) Show that for any eigenstate Ψ of H also the complex conjugate Ψ^* is an eigenstate and so are $\Re(\Psi)$ and $\Im(Psi)$.

(b) Let Ψ^G be the ground state, i.e. the eigenstate with the smallest eigenvalue of H. Show that one can find a C such that $C\Psi^G$ is everywhere real and positive.

Exercise 2: Consider a particle moving in a one-dimensional harmonic oscillator potential given by:

$$V(x) = \omega^2 x^2$$

where $\omega \in \mathbb{R}^+$. The time-independent Schrödinger equation for this system is:

$$i\frac{d}{dt}\psi = \left(-\frac{d^2}{dx^2} + V\right)\psi = H\psi$$

To find the eigenvalues of H, we introduce the dimensionless ladder operators (lowering and raising operators), defined respectively as:

$$\hat{a} = \sqrt{\omega}\left(\hat{x} + \frac{i\hat{p}}{\omega}\right), \quad \hat{a}^\dagger = \sqrt{\omega}\left(\hat{x} - \frac{i\hat{p}}{\omega}\right)$$

where $\hat{p} = -i\frac{d}{dx}$.

(a) Show that the Hamiltonian can be rewritten in terms of the ladder operators as:

$$H = \omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$$

(b) Show, that the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$ holds.

(c) Show that $Ce^{-\omega x^2}$ (with normalization factor C) is the ground state of the Harmonic oscillator. Calculate its energy.

(d) Show that for any energy-eigenstate η also $\hat{a}^\dagger\eta$ is an energy eigenstate. What is the energy difference between those two states?

(e) What does the result of (d) imply for the energy eigenvalues of H

Exercise 3: Let Ψ_t be a solution of the one-particle time-dependent Schrödinger equation with an external field:

$$i\frac{d}{dt}\Psi_t = H\Psi_t, \quad \text{where } H = -\frac{\Delta}{2} + V(x)$$

for a real-valued potential $V : \mathbb{R}^3 \to \mathbb{R}$.

Show that for any time-independent self-adjoint operator A:

(a) $\frac{d}{dt}\langle\Psi_t, A\Psi_t\rangle = \langle\Psi_t, -i[H, A]\Psi_t\rangle$

(b) If $A = x$ (the position operator), show that:

$$\frac{d}{dt}\langle\Psi_t, x\Psi_t\rangle = \langle\Psi_t, i\nabla\Psi_t\rangle$$

(c) Using the momentum-like operator from part (b), show that:

$$\frac{d}{dt}\langle\Psi_t, i\nabla\Psi_t\rangle = \langle\Psi_t, -\nabla V(x)\Psi_t\rangle$$

Compare the results of parts **(b)** and **(c)** to the equations of motion for a classical (Newtonian) particle moving in a potential $V(x)$.

Please submit the exercise sheet in pairs or groups of three via URM by 2:00 PM on July 10^{th}, 2026.