

GEOMETRY IN PHYSICS

Homework Assignment # 1

Problem 1: The sphere as a differentiable manifold

Find a C^1 -atlas for the unit sphere in \mathbb{R}^3 , i.e. for

$$S^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\}.$$

Hint: You need at least two charts. You can use, for example, spherical coordinates or the stereographic projection.

Problem 2: Equivalent atlases and differentiable structures

- (a) Prove that two atlases \mathcal{A}_1 and \mathcal{A}_2 on a topological manifold M are equivalent if and only if the identity map

$$\text{Id} : (M, [\mathcal{A}_1]) \rightarrow (M, [\mathcal{A}_2]), \quad x \mapsto x,$$

is a diffeomorphism with respect to the differentiable structures $[\mathcal{A}_1]$ and $[\mathcal{A}_2]$.

Hint: There is not much to do here beyond understanding and somewhat rephrasing the definitions of the involved notions.

- (b) Find two atlases \mathcal{A}_1 and \mathcal{A}_2 for \mathbb{R} that are not equivalent! With the atlases you found, are $M = (\mathbb{R}, [\mathcal{A}_1])$ and $N = (\mathbb{R}, [\mathcal{A}_2])$ diffeomorphic, i.e. does there exist a diffeomorphism $\phi : M \rightarrow N$?

*Hint: Start by looking for a differentiable homeomorphism $f : \mathbb{R} \rightarrow \mathbb{R}$ that is **not** a diffeomorphism.*

Problem 3: Induced differentiable structure*

Let M be a differentiable manifold, N a topological space, and $f : M \rightarrow N$ a homeomorphism. Show that N is a topological manifold and that there exists a unique differentiable structure on N such that f is a diffeomorphism.

Please hand in your written solutions on Tuesday, October 23, at the beginning of the lecture. To be admitted to the exam for this course, you need to hand in sensible solutions to **at least** half of the problems not marked with a *. Note that the problems marked with a * are not necessarily more difficult and that we strongly recommend that you try to solve all the problems on each homework assignment. If the solution to a problem is considerably more difficult or lengthy than usually, this will be indicated explicitly.