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Geometry in Physics

Homework Assignment # 10

Problem 40: Commuting flows

Let $X, Y \in \mathcal{T}^1_0(M)$ be complete vector fields and Φ^X_t and Φ^Y_s the corresponding flows. Show that the following assertions are equivalent:

(i) $[X, Y] = 0$

(ii) $\Phi^{X*}_t Y = Y$ for all $t \in \mathbb{R}$ and $\Phi^{Y*}_s X = X$ for all $s \in \mathbb{R}$

(iii) $\Phi^X_s \circ \Phi^Y_t = \Phi^Y_t \circ \Phi^X_s$ for all $s, t \in \mathbb{R}$.

Hint: Recall lemma 8.3, lemma 8.10, and proposition 7.13 from the lecture notes.

Problem 41: Hamiltonian flows and Liouville's theorem

Let $\omega \in \Lambda_2(M)$ be non-degenerate and closed. Such an ω is called a symplectic form. Show that for any $H \in C^\infty(M)$ there exists a unique vector field $X_{H,\omega}$ such that

$$\omega(X_{H,\omega}, \cdot) = \mathrm{d}H(\cdot)\,,$$

i.e. $i_{X_{H,\omega}}\omega = \mathrm{d}H$. The vector field $X_{H,\omega}$ is called the Hamiltonian vector field of the Hamiltonian function H with respect to the symplectic form ω.

Now assume that $X_{H,\omega}$ is complete. Show that the symplectic form ω is invariant under the corresponding Hamiltonian flow, i.e. that

$$\Phi^{X_{H,\omega}*}_t \omega = \omega \qquad \text{for all } t \in \mathbb{R}\,.$$

This statement is called Liouville's theorem and it shows that the flow maps of a Hamiltonian vector field are canonical transformations.

Hint: For the uniqueness of $X_{H,\omega}$ the non-degeneracy of ω is important. Liouville's theorem follows easily by taking a derivative and applying Cartan's formula (cf. the proof of lemma 8.10 in the lecture notes).

Problem 42: The tangent and the normal bundle of submanifolds

Let M be an n-dimensional manifold, N a p-dimensional manifold, $f : N \to M$ an embedding, and write $\tilde{N} := f(N) \subset M$.

(a) Show that the tangent bundle of N in M given by $T\tilde{N} := Df(TN) \subset TM|_{\tilde{N}}$ is really a subbundle of $TM|_{\tilde{N}}$ by providing explicit local trivialisations in terms of charts φ for N.

(b) Now assume in addition that there exists a smooth function $F : M \to \mathbb{R}^{n-p}$ such that $\tilde{N} = \{x \in M \mid F(x) = 0\}$ and that $DF|_x$ has full rank for all $x \in \tilde{N}$. Show that

$$T\tilde{N} = \{(x, v) \in TM|_{\tilde{N}} \mid v \in \ker(DF|_x)\}\,.$$

(c) Let $g \in \mathcal{T}^0_2(M)$ be a Riemannian metric. Formulate a definition of the normal bundle $T\tilde{N}^\perp$ in such a way that $T\tilde{N} \oplus T\tilde{N}^\perp = TM|_{\tilde{N}}$ and such that for all $v \in T_x\tilde{N}$ and $w \in T_x\tilde{N}^\perp$ one has $g|_x(v, w) = 0$.

We wish you a nice holiday and a happy new year!