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Geometry in Physics

Homework Assignment # 11

Problem 43: Subbundles of $S^1 \times \mathbb{R}^2$

(a) Show that the set of rank-1 subbundles of the trivial bundle $S^1 \times \mathbb{R}^2$ is in one-to-one correspondence with the set of smooth functions $f : \mathbb{R} \to \mathbb{R}$ that are 2π-periodic modulo π, i.e.

$$f(\varphi + 2\pi) = f(\varphi) + w \cdot \pi$$

for all $\varphi \in \mathbb{R}$ and some $w \in \mathbb{Z}$, and that satisfy $f(0) \in [0, \pi)$.

(b) Construct a rank-1 subbundle of the trivial bundle $S^1 \times \mathbb{R}^2$ that has the structure of the Möbius strip. Show that every section of this bundle has at least one zero and conclude that this bundle is not trivialisable.

(c) Two bundles over the same manifold are called isomorphic, if there exists a diffeomorphism between them that acts fibre-wise as a vector space isomorphism. Find all isomorphism classes of rank-1 subbundles of $S^1 \times \mathbb{R}^2$.

Hint: Define the "winding number" of such a subbundle and understand which winding numbers lead to isomorphic bundles. As a first step use that, according to proposition 9.18, two bundles over the same manifold that both admit a global frame are isomorphic.

Problem 44: A connection on $S^1 \times \mathbb{R}$ without non-trivial constant sections

Think of the smooth functions $C^\infty(S^1)$ on $\mathbb{R}$ as sections of the trivial line bundle $S^1 \times \mathbb{R}$. Find a connection ∇ on $S^1 \times \mathbb{R}$ such that for $f \in \Gamma(S^1 \times \mathbb{R})$ it holds that

$$\nabla_X f = 0 \text{ for some } X \in \mathcal{T}_0^1(S^1) \text{ with } X(x) \neq 0 \text{ for all } x \in S^1 \quad \Rightarrow \quad f \equiv 0\,.$$

Problem 45: The induced connection on a subbundle

Let $\pi : E \to M$ be a vector bundle with connection ∇^E. Let $F \subset E$ be a subbundle and $P_F \in \mathrm{End}(E)$ with $P_F(x)$ being a projection onto F_x for each $x \in M$. Show that

$$\nabla^F : \mathcal{T}_0^1(M) \times \Gamma(F) \to \Gamma(F)\,, \qquad (X, S) \mapsto \nabla^F_X S := P_F \nabla^E_X S\,,$$

defines a connection on F.

Given an inner product $\langle \cdot, \cdot \rangle_E$ on E, the canonical choice for $P_F(x)$ is the orthogonal projection onto the subspace F_x. A connection ∇^E on such a vector bundle is called **metric** if for all $S, T \in \Gamma(E)$ and $X \in \mathcal{T}_0^1(M)$ it holds that

$$\mathrm{d}\langle S, T \rangle_E(X) = \langle \nabla^E_X S, T \rangle_E + \langle S, \nabla^E_X T \rangle_E\,.$$

Let ∇^E be a metric connection on the bundle E with inner product $\langle \cdot, \cdot \rangle_E$ and ∇^F the canonical induced connection on a subbundle $F \subset E$. Show that ∇^F is metric with respect to the induced inner product on F.