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Geometry in Physics

Homework Assignment # 12

Problem 46: The action of affine connections on 1-forms w.r.t. coordinates

Let ∇ be an affine connection. Show that the action of its canonical lift to 1-forms is given in local coordinates by

$$\nabla_X \omega = \left(X^i \partial_i \omega_j - X^i \omega_k \Gamma^k_{ij} \right) \mathrm{d}q^j \,.$$

Problem 47: Locality of connections

Let ∇ be a connection on a vector bundle $\pi : E \to M$, $X \in \mathcal{T}^1_0(M)$, $x \in M$, and $u : (-\varepsilon, \varepsilon) \to M$ a smooth curve with $u(0) = x$ and $\dot{u}(0) = X(x)$ for some $\varepsilon > 0$.

Let $S, \tilde{S} \in \Gamma(E)$ be such that $S \circ u = \tilde{S} \circ u$. Show that $(\nabla_X S)(x) = (\nabla_X \tilde{S})(x)$.

Hence, $(\nabla_X S)(x)$ depends only on the values of S along a curve with velovity $X(x)$ through x.

Hint: Use the properties of D_t.

Problem 48: Induced connection on a submanifold

Let $\tilde{\nabla}$ be a connection on a vector bundle $\pi : E \to M$ and let $N \subset M$ be a submanifold. For $X \in \mathcal{T}^1_0(N)$ and $S \in \Gamma(E|_N)$ set

$$\nabla_X S := (\tilde{\nabla}_{\tilde{X}} \tilde{S})|_N$$

for some extensions $\tilde{X} \in \mathcal{T}^1_0(M)$ and $\tilde{S} \in \Gamma(E)$ of X resp. S. Show that $\nabla_X S$ does not depend on the choices for $\tilde{X}$ and $\tilde{S}$ and that it defines a connection on $E|_N$.

Hint: Use the statement of problem 47. Also note that you may assume that such extensions $\tilde{X}$ and $\tilde{S}$ always exist, although the proof of this assertion in not difficult and you might want to think about it, too.

Problem 49: The tangential connection of a submanifold of $\mathbb{R}^n$

Let $\tilde{\nabla}$ be the trivial connection on the (trivial) tangent bundle of $M = \mathbb{R}^n$ and let $\tilde{g} \in \mathcal{T}^0_2(M)$ denote the euclidian metric. Show that $\tilde{\nabla}$ is compatible with $\tilde{g}$, i.e. that

$$\tilde{\nabla}_{\tilde{X}} \tilde{g}(\tilde{Y}, \tilde{Z}) = \tilde{g}(\tilde{\nabla}_{\tilde{X}} \tilde{Y}, \tilde{Z}) + \tilde{g}(\tilde{Y}, \tilde{\nabla}_{\tilde{X}} \tilde{Z}) \qquad \text{for all } \tilde{X}, \tilde{Y}, \tilde{Z} \in \mathcal{T}^1_0(M)\,.$$

Let now $N \subset M$ be a submanifold and define the induced connection ∇ on TN by first restricting $\tilde{\nabla}$ to $TM|_N$ as in problem 48 and then restricting the latter to the subbundle $TN \subset TM|_N$ as in problem 45 using the orthogonal projection in $\mathbb{R}^n$ with respect to the euclidian metric.

Show that ∇ is compatible with respect to the restriction g of $\tilde{g}$ to N, i.e. that

$$\nabla_X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \qquad \text{for all } X, Y, Z \in \mathcal{T}^1_0(N)\,.$$