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Geometry in Physics

Homework Assignment # 13

Problem 50: Christoffel symbols for the tangential connection on S^2

Consider the sphere S^2 as a submanifold of $\mathbb{R}^3$, i.e. $S^2 := \{x \in \mathbb{R}^3 \,|\, \|x\| = 1\} \subset \mathbb{R}^3$. A local chart is given by the inverse of the map

$$\varphi^{-1} : (0,\pi) \times (0,2\pi) \to V \subset S^2 \subset \mathbb{R}^3\,, \quad (\theta,\phi) \mapsto (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\,,$$

where $V := \varphi^{-1}((0,\pi) \times (0,2\pi))$ is the domain of the chart.

Compute all Christoffel symbols of the tangential connection on S^2 (c.f. problem 49) with respect to the coordinate vector fields ∂_ϕ and ∂_θ on S^2 by directly computing e.g.

$$\nabla_{\partial_\phi}\partial_\theta := P(\theta,\phi)\tilde{\nabla}_{\tilde{\partial}_\phi}\tilde{\partial}_\theta$$

for appropriate extensions $\tilde{\partial}_\phi$ and $\tilde{\partial}_\theta$ of the coordinate vector fields to an open neighbourhood of $V \subset \mathbb{R}^3$ and $P(\theta,\phi)$ the orthogonal projection onto the tangent space of S^2 at the point (θ,ϕ).

Hint: You can extend φ to a neighbourhood $\tilde{V}$ of V by setting

$$\tilde{\varphi}^{-1} : (0,\infty) \times (0,\pi) \times (0,2\pi) \to \tilde{V} \subset \mathbb{R}^3\,, \quad (r,\theta,\phi) \mapsto r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\,.$$

Problem 51: Parallel transport along curves in $\mathbb{R}^3$

Let $u : \mathbb{R} \to \mathbb{R}^n$ be a smooth curve of unit speed, i.e. such that $\langle \dot{u}(t), \dot{u}(t)\rangle_{\mathbb{R}^n} \equiv 1$. Assume, for simplicity, that u is an embedding, i.e. that $U := u(\mathbb{R}) \subset \mathbb{R}^n$ is a one-dimensional submanifold.

The restriction of the tangent bundle $T\mathbb{R}^n$ to the submanifold U splits into the orthogonal sum of the tangent bundle TU and the normal bundle NU, i.e. $T\mathbb{R}^n|_U = TU \oplus NU$. Show that a section $\tilde{v} \in \Gamma(NU)$ of the normal bundle NU is parallel with respect to the connection ∇^N induced by the trivial connection ∇ on $T\mathbb{R}^n$, if and only if $v : \mathbb{R} \to \mathbb{R}^n$, $t \mapsto (\tilde{v} \circ u)(t)$ satisfies the differential equation

$$\dot{v}(t) = -\langle \ddot{u}(t), v(t)\rangle_{\mathbb{R}^n}\, \dot{u}(t)\,.$$

Hint: Understand that $D^N_t v = P_{N_{u(t)}U} D_t v = P_{N_{u(t)}U}\dot{v}(t)$, where $P_{N_{u(t)}U}$ denotes the orthogonal projection onto the normal space at $u(t)$, D^N_t the covariant derivative along u induced by ∇^N, and D_t the covariant derivative along u induced by ∇.

Problem 52: Again the tangential connection

As in problem 49, let $\tilde{\nabla}$ be the trivial connection on the (trivial) tangent bundle of $M = \mathbb{R}^n$, $\tilde{g} \in \mathcal{T}^0_2(M)$ the euclidian metric, and $N \subset M$ a submanifold. Denoting by $\psi : N \hookrightarrow M$ the injection, we can define the restriction of $\tilde{g}$ to N by $g := \psi^*\tilde{g}$. Show that the Levi-Civita connection of g agrees with the tangential connection on N constructed in problem 49.

Hint: Because of the uniqueness of the Levi-Civita connection it suffices to prove symmetry of the tangential connection and combine it with known results.