Stefan Teufel and Giovanna Marcelli
Mathematisches Institut, Universität Tübingen

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Geometry in Physics

Homework Assignment # 14

Problem 53: Again the Christoffel symbols for the tangential connection on S^2

Consider $S^2 \subset \mathbb{R}^3$ with spherical coordinates as in problem 50. Compute the coordinate representation of the induced metric g on S^2 with respect to spherical coordinates and then the Christoffel symbols of the Levi-Civita connection using the formula of theorem 11.7 in the lecture notes. Compare with the result of problem 50.

Problem 54: Naturality of the exponential map

Let $\Phi : M \to \tilde{M}$ be an isometry of (pseudo-)Riemannian manifolds (M, g) and $(\tilde{M}, \tilde{g})$ and denote by exp and $\widetilde{\exp}$ the exponential maps with respect to the corresponding Riemannian connections. Show that

$$\Phi \circ \exp_x = \widetilde{\exp}_{\Phi(x)} \circ \Phi_*|_{\mathcal{E}_x} .$$

Problem 55: The Ricci tensor

Show that the Ricci tensor is symmetric and can be expressed in the following ways:

$$\mathrm{Ric}_{ij} = R_{kij}{}^{k} = R_{ik}{}^{k}{}_{j} = -R_{ki}{}^{k}{}_{j} = -R_{ikj}{}^{k} .$$

Problem 56: The curvature tensors of the sphere and of the cylinder

Consider the sphere $S^2 \subset \mathbb{R}^3$ with spherical coordinates and the cylinder

$$C := \{x \in \mathbb{R}^3 \,|\, x_1^2 + x_2^2 = 1\} \subset \mathbb{R}^3$$

with coordinates $\varphi^{-1} : [0, 2\pi) \times \mathbb{R} \to C$, $(\alpha, z) \mapsto (\cos\alpha, \sin\alpha, z)$. Compute in both cases the components of the curvature tensor of the tangential connection with respect to the corresponding coordinates. Then compute also the Ricci tensor and the scalar curvature.

Problem 57: The covariant Hessian $*$

Let ∇ be an affine connection on M and let $u \in C^\infty(M)$, $\omega \in \Lambda_1(M)$, and $X, Y \in \mathcal{T}_0^1(M)$. Then the second covariant derivative $\nabla^2 u := \nabla(\nabla u) \in \mathcal{T}_2^0(M)$ is called the covariant Hessian of u and $\nabla^2\omega := \nabla(\nabla\omega) \in \mathcal{T}_3^0(M)$ the covariant Hessian of ω. Show that

$$\nabla^2 u(X, Y) = L_X L_Y u - L_{\nabla_X Y} u$$

and

$$\nabla^2\omega(X, Y, Z) = (\nabla_X \nabla_Y \omega)(Z) - (\nabla_{\nabla_X Y}\omega)(Z)$$

Now assume that ∇ is a symmetric connection. Show that the covariant Hessian of a function u is a symmetric $(0, 2)$-tensor, i.e. that

$$\nabla^2 u(X, Y) = \nabla^2 u(Y, X)\,,$$

and that the covariant Hessian of a 1-form ω satisfies the *Ricci identity*

$$\nabla^2\omega(X, Y, Z) - \nabla^2\omega(Y, X, Z) = \omega(\mathcal{R}(X, Y)Z)\,.$$

Problem 58: The curvature 2-form $*$

Let ∇ be a connection on the vector bundle $\pi : E \to M$, $\omega \in \Gamma(T^*M \otimes \mathrm{End}(E))$ an endomorphism-valued 1-form, and $\tilde{\nabla} := \nabla + \omega$ (cf. proposition 10.11). Argue that the curvature map of a connection can be viewed as an endomorphism-valued 2-form $\mathcal{R} \in \Gamma(T^*M \otimes T^*M \otimes \mathrm{End}(E))$ by defining $\mathcal{R} : (X, Y) \mapsto \mathcal{R}(X, Y, \cdot)$. Show that

$$\widetilde{\mathcal{R}}(X, Y) = \mathcal{R}(X, Y) \,+\, \mathrm{d}\omega(X, Y) \,+\, [\omega(X), \omega(Y)]\,,$$

where the differential $\mathrm{d}\omega$ of a "vector-valued" one-form is defined as in the case of real-valued one-forms, i.e. in local coordinates $\mathrm{d}\omega = \omega_{i,j}\,\mathrm{d}q^j \wedge \mathrm{d}q^i$ with endomorphism-valued coefficient functions ω_i.

Hint: Use local coordinates to show that

$$\mathrm{d}\omega(X, Y) = [\nabla_X, \omega(Y)] - [\nabla_Y, \omega(X)] - \omega([X, Y])\,.$$