Stefan Teufel and Giovanna Marcelli
Mathematisches Institut, Universität Tübingen

Winter Semester 2018/19
23.10.2018

Geometry in Physics

Homework Assignment # 2

Problem 4: Tangent vectors as derivations

In the lecture it was shown that every tangent vector $v \in T_xM$ to a manifold M at a point $x \in M$ defines a derivation D_v at that point. In this problem we motivate that these are actually all derivations at a point $x \in M$, i.e. that every derivation D at x is of the form $D = D_v$ for some tangent vector $v \in T_xM$, by proving the corrsponding statement for $M = \mathbb{R}^n$.

Recall that for a n-dimensional smooth manifold M a **derivation at** $x \in M$ is a linear map $D : C^\infty(M, \mathbb{R}) \to \mathbb{R}$ that satisfies the product rule

$$D(fg) = D(f)\, g(x) + f(x)\, D(g) \qquad \text{for all } f, g \in C^\infty(M, \mathbb{R})\,. \tag{1}$$

Now let $M = \mathbb{R}^n$. Prove that every derivation D at $x \in \mathbb{R}^n$ is of the form D_v for some tangent vector $v \in T_xM \cong \mathbb{R}^n$, where

$$D_v(f) = v \cdot \nabla f(x)\,. \tag{2}$$

Hint: First show that (1) *implies that* $D(1) = 0$, *where* $1 \in C^1(M)$ *denotes the function that is constant equal to one. Conclude that* $D(f) = 0$ *for any constant function. Now look at the first order Taylor expansion of* f *at the point* x *with explicit remainder term in integral form to conclude that* $D(f)$ *is of the form* (2).

Problem 5: Polar coordinates

We equip the manifold $M = \mathbb{R}^2 \setminus \{x \in \mathbb{R}^2 \,|\, x_2 = 0 \text{ and } x_1 \leq 0\} \subset \mathbb{R}^2$ with the charts (M, φ_1) and (M, φ_2) where $\varphi_1(x) = (x_2, x_1)$ and

$$\varphi_2^{-1} : (0, \infty) \times (-\pi, \pi) \to M\,, \quad (r, \theta) \mapsto (r\cos\theta, r\sin\theta)\,.$$

Show that φ_1 and φ_2 are compatible and express the coordinate vectors ∂_{q_1} and ∂_{q_2} associated with φ_1 as well as the coordinate vectors ∂_r and ∂_θ associated with φ_2 at a point $x \in M$ in terms of the canonical basis vectors e_1 and e_2 in $T_xM \cong \mathbb{R}^2$.

Problem 6: Inverse function theorem on manifolds

Let M and N be differentiable manifolds of dimension n and $f \in C^1(M, N)$. Assume that at some $x \in M$ the differential $Df|_x : T_xM \to T_{f(x)}N$ is an isomorphism. Use the inverse function theorem on $\mathbb{R}^n$ in order to show that there exists an open neighbourhood $U \subset M$ of x, such that $V := f(U)$ is open and $f : U \to V$ is a diffeomorphism.

Problem 7: Projectives spaces*

The real projective space $P^n(\mathbb{R})$ is defined as the quotient

$$P^n(\mathbb{R}) := (\mathbb{R}^{n+1} \setminus \{0\})/ \sim$$

with respect to the equivalence relation

$$x \sim y \quad :\Leftrightarrow \quad \text{there exists } \lambda \in \mathbb{R} \text{ such that } x = \lambda y \,.$$

Find an atlas for $P^2(\mathbb{R})$ of charts that are homeomorphisms with respect to the quotient topology on $P^2(\mathbb{R})$ and that turns $P^2(\mathbb{R})$ into a smooth two-dimensional manifold.

Hint: A point in $P^2(\mathbb{R})$ is a one-dimensional linear subspace of $\mathbb{R}^3$.

Please hand in your written solutions on Tuesday, October 30, at the beginning of the lecture. To be admitted to the exam for this course, you need to hand in sensible solutions to **at least** half of the problems not marked with a $*$. Note that the problems marked with a $*$ are not necessarily more difficult and that we strongly recommend that you try to solve all the problems on each homework assignement. If the solution to a problem is considerably more difficult or lengthy than usually, this will be indicated explicitly.