Stefan Teufel and Giovanna Marcelli
Mathematisches Institut, Universität Tübingen

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Geometry in Physics

Homework Assignment # 3

Problem 8: Level sets of functions as submanifolds

Let M and N be smooth manifolds of dimensions $m > n$ respectively. Let $F : M \to N$ be a smooth map and $y \in N$ a regular value for F. A value $y \in N$ of a function $F : M \to N$ is called **regular**, if the differential $DF|_{T_xM}$ has full rank equal to n for all $x \in F^{-1}(\{y\}) \subset M$.

Show that the level set $F^{-1}(\{y\})$ is a $(m-n)$-dimensional submanifold of M.

Hint: Proposition 2.21 (c) from the lecture notes.

Problem 9: The orthogonal matrices as a manifold

Show that the orthogonal matrices $O(n) := \{Q \in \mathrm{GL}(n) \,|\, Q^TQ = \mathrm{Id}\}$ form a $\frac{n(n-1)}{2}$-dimensional submanifold of the manifold of $n \times n$-matrices $\mathrm{mat}(n) \cong \mathbb{R}^{n^2}$. Show also that

$$T_QO(n) = \{B \in \mathrm{mat}(n) \,|\, (Q^{-1}B)^T = -Q^{-1}B\}\,,$$

and hence, in particular,

$$T_{\mathrm{Id}}O(n) \;=\; \{B \,|\, B^T = -B\} \;=:\; \mathrm{skew}(n).$$

Hint: Find a suitable map $F : \mathrm{mat}(n) \to \mathrm{sym}(n)$ *into the symmetric matrices sym*(n) *with* $F^{-1}(\{0\}) = O(n)$ *and apply problem 8.*

Problem 10: The tangent bundle

Let M be a n-dimensional differentiable manifold and TM its tangent bundle.

(a) Show that M is diffeomorphic to a submanifold of TM.

(b) Show that TM is trivializable if and only if there exist n vector fields on M that are pointwise linear independent.

Problem 11: The Lie derivative of functions

Let M be a differentiable manifold. Show the following properties of the Lie-derivative:

(i) $L_X(f+g) = L_Xf \,+\, L_Xg$

(ii) $L_X(fg) = (L_Xf)g \,+\, f(L_Xg)$

(iii) $L_{\alpha X+\beta Y}(f) = \alpha\, L_Xf \,+\, \beta\, L_Yf$

for all $f, g, \alpha, \beta \in C^\infty(M)$ and $X, Y \in \mathcal{T}^1_0(M)$.

Problem 12: The commutator of vector fields $*$

Let M be a differentiable manifold and $X, Y \in \mathcal{T}_0^1(M)$. Show that there exists a vector field $Z \in \mathcal{T}_0^1(M)$ such that

$$L_X \circ L_Y \; - \; L_Y \circ L_X \; = \; L_Z .$$

Hint: You may either use the statement of remark 2.32 in the lecture notes or do an explicit computation using charts. In the latter approach you need to show, however, that the vector field Z does not depend on the choice of charts.

Please hand in your written solutions on Tuesday, November 6, before the beginning of the lecture.