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Geometry in Physics

Homework Assignment # 4

Problem 12: Change of basis

Let V be a n-dimensional vector space and V^* its dual. Let $(e_j)_{j=1,\dots,n}$ be a basis of V and $(e^j)_{j=1,\dots,n}$ the dual basis of V^* defined by $e^j(e_i) := \delta_{ij}$. Let $A : V \to V$ be an isomorphism with matrix (a^i_j) w.r.t. the basis $(e_j)_{j=1,\dots,n}$ and $\hat{e}_j := Ae_j$ its image under A, i.e.

$$\hat{e}_j \;=\; \sum_i a^i_j \, e_i .$$

What is the transformation law for the dual basis? More precisely: Let $(\hat{e}^j)_{j=1,\dots,n}$ be the dual basis for $(\hat{e}_j)_{j=1,\dots,n}$. Determine the matrix (b^j_k), such that

$$\hat{e}^j \;=\; \sum_k b^j_k \, e^k .$$

Verify your result by applying it to the coordinate bases $(\partial_r, \partial_\theta)$ and $(\mathrm{d}r, \mathrm{d}\theta)$ on $\mathbb{R}^2$ resp. $\mathbb{R}^{2*}$ with respect to polar coordinates (r, θ), cf. problem 5 from the second assignment and example 3.15 from the lecture notes.

Problem 13: Line integrals of 1-forms

Let M be a smooth manifold, $I = [a, b] \subset \mathbb{R}$ an interval, $\gamma \in C^\infty(I, M)$ a smooth curve, and $\omega \in \mathcal{T}^0_1(M)$ a 1-form. Then the integral of ω along γ is the number

$$\int_\gamma \omega := \int_I \gamma^*\omega := \int_a^b (\gamma^*\omega \,|\, e)(t)\, \mathrm{d}t ,$$

where $\gamma^*\omega$ is the pull-back of ω to I under γ and $e : I \to I \times \mathbb{R}$, $t \mapsto (t, 1)$, is the unit vector field on I. The dual pairing $(\gamma^*\omega \,|\, e) \in C^\infty(I)$ between $\gamma^*\omega \in \mathcal{T}^0_1(I)$ and $e \in \mathcal{T}^1_0(I)$ is to be taken pointwise and defines a smooth function on I that is integrated in the standard Riemannian sense.

(a) For $t \in I$ let $\gamma'(t) := D\gamma|_{T_t I}\, e(t) \in T_{\gamma(t)}M$. Show that

$$\int_\gamma \omega = \int_a^b \big(\omega(\gamma(t)) \,|\, \gamma'(t)\big)\, \mathrm{d}t . \qquad (*)$$

(b) Let $\tilde{I} = [\tilde{a}, \tilde{b}]$ be another open interval, $\Phi : \tilde{I} \to I$ a diffeomorphism with $\Phi'(t) > 0$, and $\tilde{\gamma} : \tilde{I} \to M$, $\tilde{\gamma}(t) := (\gamma \circ \Phi)(t)$, a reparametrisation of the curve γ. Show that

$$\int_{\tilde{\gamma}} \omega = \int_\gamma \omega .$$

Hint: First show that $\tilde{\gamma}'(t) = \Phi'(t)\, \gamma'(\Phi(t))$ and then use $(*)$ and the usual substitution rule for one-dimensional integrals.

(c) Now let $f \in C^\infty(M)$ and $\mathrm{d}f \in \mathcal{T}^0_1(M)$ its differential. Prove the fundamental theorem of calculus:

$$\int_\gamma \mathrm{d}f = f(\gamma(b)) - f(\gamma(a)) .$$

Hint: First show that $\big(\mathrm{d}f(\gamma(t)) \,|\, \gamma'(t)\big) = \frac{\mathrm{d}}{\mathrm{d}s} f(\gamma(s))\big|_{s=t}$ and then use the fundamental theorem of calculus for functions on $\mathbb{R}$.

Problem 14: Line integrals continued*

According to problem 13 (c) the line integral of a differential depends only on the end-points of the curve γ. Now assume that a 1-form $\omega \in \mathcal{T}^0_1(M)$ is given on a connected manifold M, such that

$$\int_\gamma \omega = \int_{\tilde\gamma} \omega \quad \text{whenever} \quad \gamma(a) = \tilde\gamma(\tilde a) \text{ and } \gamma(b) = \tilde\gamma(\tilde b)\,.$$

I.e., the line integral of ω along any curve γ depends only on the end-points of γ.

Show that there exists a function $f \in C^\infty(M)$ such that $\omega = \mathrm{d}f$.

Hint: Construct f explicitly by integration. You may assume without proof that for any two points $x, y \in M$ and any given smooth curve through x the latter may be smoothly extended to a curve that passes also through y.

Problem 15: Heat and work as 1-forms

For a system with a fixed number of particles one can characterise a thermal equilibrium state by specifying the volume $V \in (0,\infty)$ and the entropy $S \in (0,\infty)$ of the system. In the following we think of the thermodynamic state space $M = (0,\infty)^2 \subset \mathbb{R}^2$ of such a system as a smooth manifold. The energy E is a function $E : M \to \mathbb{R}$, $(S,V) \mapsto E(S,V)$, on the space of equilibrium states.

Show that the differential $\mathrm{d}E \in \mathcal{T}^0_1(M)$ has the following representation with respect to the coordinate 1-forms $\mathrm{d}S$ and $\mathrm{d}V$,

$$\mathrm{d}E = \frac{\partial E}{\partial S}\,\mathrm{d}S + \frac{\partial E}{\partial V}\,\mathrm{d}V =: T\,\mathrm{d}S \;-\; p\,\mathrm{d}V\,.$$

To do so, work out the details to show (3.2) in remark 3.13 in the lecture notes and apply the result to the present case.

Physics background: The 1-form $\delta Q := T\,\mathrm{d}S$ is called the heat (absorbed by the system) and the 1-form $\delta W := -p\,\mathrm{d}V$ is called the work (performed on the system). Thus, in contrast to the energy E, the volume V, the entropy S, the temperature T, and the pressure p, heat and work are not functions of the equilibrium state, but 1-forms on the manifold of equilibrium states.

Hence, it makes only sense to ask for the heat transferred resp. work performed during a specific thermodynamic process, i.e. for the integrals of the corresponding 1-forms along a curve in the space of equilibrium states. Since, in general, δQ and δW are **not** differentials of functions, transferred heat and performed work during a process depend not only on the initial and final state but on the specific path. In contrast, the change of energy, i.e. the integral of $\mathrm{d}E$, is just the difference of the final and initial energies and vanishes for a closed cycle. However, the integrals of δQ and δW typically do not vanish for closed cycles, a fact that makes possible the construction of heat engines.

Please hand in your written solutions on Tuesday, November 13, at the beginning of the lecture.