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Geometry in Physics

Homework Assignment # 5

Problem 16: The differential of the determinant

Consider the group $GL(n,\mathbb{R})$ of invertible $n\times n$-matrices with the matrix-entries A_i^j as coordinates (i.e. as an open subset of $\mathbb{R}^{n^2}$) and the smooth map

$$\det : GL(n,\mathbb{R}) \to \mathbb{R}\,, \quad A \mapsto \det A := \text{determinant of } A\,.$$

(a) Show that the partial derivatives of the determinant map are given by

$$\frac{\partial}{\partial A_i^j} \det A = (\det A)\,(A^{-1})_j^i\,.$$

Hint: Use Laplace's expansion of $\det A$ *by minors along the* i*-th column and then use Cramer's rule.*

(b) Conclude that the differential $\mathrm{d}\det|_A$ of the determinant function at the point $A \in GL(n,\mathbb{R})$ acts on a tangent vector $B \in T_A GL(n,\mathbb{R}) \cong \mathrm{mat}(n,\mathbb{R}) \cong \mathbb{R}^{n^2}$ as

$$(\mathrm{d}\det|_A \,|\, B) = (\det A)\,\mathrm{tr}(A^{-1}B)\,.$$

Here $\mathrm{tr}(C) := \sum_{i=1}^n C_i^i$ denotes the trace of $C \in \mathrm{mat}(n,\mathbb{R})$.

Problem 17: Isometries of Minkowski spacetime 1

On $V = \mathbb{R}^4$ let $\eta \in V_2^0$ be the Minkowski metric, i.e. $\eta_{ij} = \sigma_i \delta_{ij}$ with $\sigma_1 = -1$ and $\sigma_{i\neq 1} = 1$.

(a) Let $\lambda \in V_1^1$. Argue that λ defines a linear map $\Lambda : V \to V$. Then show that Λ is an isometry, i.e. $\eta(\Lambda v, \Lambda v) = \eta(v,v)$ for all $v \in V$, if and only if

$$\lambda_k^i \eta_{ij} \lambda_\ell^j = \eta_{k\ell} \qquad \text{i.e.} \qquad \Lambda^T \eta\, \Lambda \;=\; \eta.$$

Argue that in this notation the matrix representing the linear map Λ is written in the form $\Lambda_{ij} = \lambda_j^i$ (that is with an upper row-index and a lower column-index).

(b) Show that an orthogonal transformation in the last three coordinates, i.e.

$$R \;=\; \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}$$

with $r \in O(3)$ as well as the s-boost in the x-direction, i.e.

$$B_x^s \;=\; \begin{pmatrix} \cosh s & \sinh s & 0 & 0 \\ \sinh s & \cosh s & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad s \in \mathbb{R}\,,$$

are isometries of Minkowski spacetime. Conclude that special Lorentz transformations of the form

$$\Lambda \;=\; R_1\, B_x^s\, R_2$$

are isometries as well.

Problem 18: Isometries of Minkowski spacetime 2 ∗

Let again $V = \mathbb{R}^4$ with the Minkowski metric η and let $\lambda \in V_1^1$ correspond to an isometric isomorphism denoted by Λ. In addition assume that λ is future preserving, i.e.

$$\eta(e_1, \Lambda e_1) \; < \; 0,$$

where $e_1 = (1, 0, 0, 0)$. Show that Λ is a special Lorentz transformation, i.e. that it can be written in the form

$$\Lambda \; = \; R_1 \, B_x^s \, R_2$$

with R_i and B_x^s as in problem 17.

Hint: Consider Λe_1 and find R_1^{-1} and B_x^s such that $R_1^{-1} \Lambda e_1 = B_x^s e_1$.

Problem 19: Constant functions and pull-backs

Let N be a smooth connected manifold and $f \in C^\infty(M)$. Show that f is constant if and only if $\mathrm{d}f \equiv 0$.

Now let $\gamma : N \to M$ be a smooth map into another manifold M. Show that a function $g \in C^\infty(M)$ is constant on the set $\gamma(N) \subset M$ if and only if $\gamma^*(\mathrm{d}g) \equiv 0$.

Please hand in your written solutions on Tuesday, November 20, at the beginning of the lecture.