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Winter Semester 2018/19
20.11.2018

Geometry in Physics

Homework Assignment # 6

Problem 21: The wedge product

Let $\omega \in \Lambda_k$ and $\nu \in \Lambda_p$. Show that

$$\omega \wedge \nu = (-1)^{kp}\, \nu \wedge \omega\,.$$

Hint: You can use basis representations and the property (5.1) of the wedge product of covectors.

Problem 22: The canonical volume form

Let $g \in V_2^0$ be non-degenerate, $(e^j)_{j=1,\dots,n}$ and $(\tilde{e}^j)_{j=1,\dots,n}$ bases of V^* with the same orientation, and

$$g = \sum_{i,j=1}^{n} g_{ij}\, e^i \otimes e^j = \sum_{i,j=1}^{n} \tilde{g}_{ij}\, \tilde{e}^i \otimes \tilde{e}^j\,.$$

Show that

$$\sqrt{|\det g_{ij}|}\, e^1 \wedge e^2 \wedge \dots \wedge e^n = \sqrt{|\det \tilde{g}_{ij}|}\, \tilde{e}^1 \wedge \tilde{e}^2 \wedge \dots \wedge \tilde{e}^n\,, \qquad (*)$$

i.e. that the canonical volume form ε defined by the expression $(*)$ does not depend on the choice of basis.

Problem 23*: The Hodge isomorphism

In this assignment we show that for a non-degenerate and symmetric $g \in V_2^0$ the Hodge operator satisfies

$$* \circ *|_{\Lambda_k} \;=\; (-1)^{k\,(n-k)}\,\mathrm{sgn}(\det g_{ij})\,. \qquad (1)$$

(a) First argue that there exists a basis (e^j) of V^* such that the component matrix g_{ij} of $g = g_{ij} e^i \otimes e^j$ is diagonal, i.e. $g_{ij} = 0$ if $i \neq j$.

(b) Let $0 \le j_1 < j_2 < \dots < j_k \le n$ be an ordered k-tuple with $j_i \in \{1, \dots, n\}$. Now compute $*(e^{j_1} \wedge \dots \wedge e^{j_k}) := i_{e^{j_1} \wedge \dots \wedge e^{j_k}}\varepsilon$, where (e^j) is the basis from (a).

(c) Finally show equation (1).

Problem 24: Hodge duality in Minkowski space

Let $*$ be the Hodge operator with respect to the Minkowski metric η on $\mathbb{R}^4$. Denoting the canonical coordinates of $\mathbb{R}^4$ with (t, x_1, x_2, x_3) we have $\eta = -\mathrm{d}t \otimes \mathrm{d}t + \sum_{i=1}^{3} \mathrm{d}x^i \otimes \mathrm{d}x^i$. Compute the images of the canonical basis vectors of $\Lambda_k(\mathbb{R}^4)$ for each $0 \le k \le 4$.

Hint: Save time by using formula (1) from problem 23!

Problem 25: Naturality of the exterior derivative

Let $f : M \to N$ be a smooth map between smooth manifolds M and N and let $\omega \in \Lambda_p(N)$. Show that

$$f^*\mathrm{d}\omega = \mathrm{d}(f^*\omega)\,.$$

Hint: Look at the computation in the proof of proposition 5.16 in the lecture notes.