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# Geometry in Physics

Homework Assignment # 7

### Problem 26: Maxwell's equations

Let $*$ be the Hodge operator with respect to the Minkowski metric $\eta$ on $\mathbb{R}^4$. We assume that the electric field $E$, the magnetic field $B$, and the current density $J$ are smooth time-dependent vector fields on $\mathbb{R}^3$, e.g.

$$E : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3 , \quad (t, x) \mapsto E(t, x) .$$

The charge density is a smooth real-valued function $\rho : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$.
One now defines corresponding differential forms on Minkowski space $\widetilde{\mathcal{J}}, \mathcal{E} \in \Lambda_1(\mathbb{R}^4)$ and $\mathcal{B}, \mathcal{F} \in \Lambda_2(\mathbb{R}^4)$ by

$$\begin{aligned} \widetilde{\mathcal{J}} &:= \iota_\eta(\rho\,\partial_t, J \cdot \partial_x) \;=\; \rho\,\mathrm{d}t \;-\; J_i\,\mathrm{d}x^i, \\ \mathcal{E} &:= E_i\,\mathrm{d}x^i, \\ \mathcal{B} &:= *\,(-B_i\,\mathrm{d}t \wedge \mathrm{d}x^i), \\ \mathcal{F} &:= \mathcal{B} - \mathrm{d}t \wedge \mathcal{E} . \end{aligned}$$

Finally we define the current 3-form as $\mathcal{J} := *\widetilde{\mathcal{J}}$. (Why is it natural to view the current as a 3-form?)
Prove the following two equivalences:

$$\frac{\partial B}{\partial t} + \mathrm{curl} E \;=\; 0 \quad \& \quad \mathrm{div} B \;=\; 0 \quad \Longleftrightarrow \quad \mathrm{d}\,\mathcal{F} \;=\; 0,$$

$$-\frac{\partial E}{\partial t} + \mathrm{curl} B \;=\; J \quad \& \quad \mathrm{div} E \;=\; \rho \quad \Longleftrightarrow \quad \mathrm{d}(*\mathcal{F}) \;=\; \mathcal{J}.$$

Thus, Maxwell's equations have a very simple form when written in terms of differential forms.

### Problem 27: The continuity equation on Minkowski space

Let $*$ be the Hodge operator with respect to the Minkowski metric $\eta$ on $\mathbb{R}^4$. Given a time-dependent smooth vector field $J : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ and a time-dependent smooth density $\rho : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$, we define the current 3-form $\mathcal{J} \in \Lambda_3(\mathbb{R}^4)$ as in problem 26. Show that continuity equation

$$\frac{\partial \rho}{\partial t} + \mathrm{div} J = 0$$

is equivalent to

$$\mathrm{d}\,\mathcal{J} \;=\; 0 .$$

Show, in addition, that the inhomogeneous Maxwell equation

$$\mathrm{d}\,{*\mathcal{F}} \;=\; \mathcal{J}$$

has a solution $\mathcal{F}$ if and only if the current $\mathcal{J}$ solves the conitnuity equation. Is the solution unique?

**Problem 28*: The homotopy operator: preparation for problem 29**

Let $M$ be a manifold and $[0,1]\times M$ the product manifold with boundary $(\{0\}\times M)\cup(\{1\}\times M)\cup((0,1)\times\partial M)$. Let $\iota_t : M\to[0,1]\times M$, $x\mapsto(t,x)$ denote the injection and $\pi:[0,1]\times M\to M$, $(t,x)\mapsto x$ the projection onto $M$.

(a) Show that every $\omega\in\Lambda_p([0,1]\times M)$ can be split as

$$\omega = \mathrm{d}t\wedge\omega_M + \omega_0\,,$$

where $\omega_M\in\Lambda_{p-1}([0,1]\times M)$ is given by $\omega_M(\cdot)=\omega(\partial_t,\cdot)$ and $\omega_0\in\Lambda_p([0,1]\times M)$ by $\omega_0\big|_{(t,\cdot)}=\pi^*\iota_t^*\omega$. To this end represent both sides of the above equation with respect to a local coordinate basis $(\mathrm{d}t,\mathrm{d}q^1,\ldots,\mathrm{d}q^n)$ where $q$ are local coordinates on $M$.

(b) One now defines the homotopy operator $K:\Lambda_p([0,1]\times M)\to\Lambda_{p-1}(M)$ by

$$\omega=\mathrm{d}t\wedge\omega_M+\omega_0\mapsto K\omega:=\int_0^1\omega_M(t)\,\mathrm{d}t\,,$$

where $\omega_M(t):=\iota_t^*\omega_M$. Show that

$$\mathrm{d}\circ K+K\circ\mathrm{d}=\iota_1^*-\iota_0^*\,.$$

**Problem 29: Poincaré lemma**

Let $\omega\in\Lambda_p(M)$ be closed and $M$ contractible, i.e. there exists a smooth map

$$F:[0,1]\times M\to M \quad\text{with}\quad F(0,\cdot)=\mathrm{id}_M \quad\text{and}\quad F(1,\cdot)\equiv x_0 \quad\text{for some } x_0\in M\,.$$

Thus $F$ "contracts" $M$ continuously into a single point $x_0\in M$. Show that $\omega$ is exact, i.e. $\omega=\mathrm{d}\nu$ for some $\nu\in\Lambda_{p-1}(M)$.

*Hint: Define $\Omega:=F^*\omega$ and act with $\mathrm{d}\circ K+K\circ\mathrm{d}$ from problem 28 (b) on $\Omega$.*

Please hand in your written solutions on Tuesday, December 4, at the beginning of the lecture.