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Geometry in Physics

Homework Assignment # 8

On this assignment you need to hand in only four out of the six problems in order to obtain 100% of the credits. In other words, you can put a $*$ on two problems of your choice.

Problem 30: The Laplace-Beltrami operator

Let M be a smooth manifold euqipped with a (pseudo-)metric $g \in \mathcal{T}_2^0(M)$ and denote by $*$ the corresponding Hodge operator. On $\Lambda_k(M)$ one can define the co-differential $\delta : \Lambda_k(M) \to \Lambda_{k-1}(M)$ by $\delta := (-1)^k *^{-1} \mathrm{d}\, *$, where according to problem 23 we have $*^{-1} = (-1)^{k(n-k)} \mathrm{sgn}(g) *$ on Λ_k.

(a) Let $M = \mathbb{R}^n$ be equipped with the euclidean metric. Show that for $f \in C^\infty(\mathbb{R}^n) = \Lambda_0(\mathbb{R}^n)$ it holds that

$$(\delta \mathrm{d} + \mathrm{d}\delta) f = -\Delta f ,$$

where $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial {q_i}^2}$ is the standard Laplacian.

(b) Let $M = \mathbb{R}^4$ be equipped with the Minkowski metric. Show that for $f \in C^\infty(\mathbb{R}^4) = \Lambda_0(\mathbb{R}^4)$ it holds that

$$(\delta \mathrm{d} + \mathrm{d}\delta) f = -\left(\frac{\partial^2}{\partial t^2} - \Delta \right) f .$$

Problem 31: Potentials and Maxwell equations

We consider once more Maxwell's equations and adopt the same notation as in problems 26, 27, and 30. Let

$$\delta : \Lambda_p(\mathbb{R}^4) \to \Lambda_{p-1}(\mathbb{R}^4) , \quad \omega \mapsto \delta\omega := (-1)^p *^{-1} \mathrm{d} * \omega = * \,\mathrm{d}\, *$$

be the co-differential, where the last equality holds specifically for $\mathbb{R}^4$ equipped with the Minkowski metric.

Assume that $\mathcal{A} \in \Lambda_1(\mathbb{R}^4)$ solves the wave equation

$$\Box \mathcal{A} := (\delta \mathrm{d} + \mathrm{d}\delta)\mathcal{A} = \widetilde{\mathcal{J}}$$

and the Lorenz gauge condition

$$\delta \mathcal{A} = 0 .$$

(a) Show that $\mathcal{F} := \mathrm{d}\mathcal{A}$ solves Maxwell's equations.

(b) Formulate the wave equation and the Lorenz gauge condition in terms of the components of the basis representation

$$\mathcal{A} = A_0 \mathrm{d}t + A_1 \mathrm{d}x^1 + A_2 \mathrm{d}x^2 + A_3 \mathrm{d}x^3 .$$

Problem 32: Integrals of closed forms for diffeotopic manifolds

Let M be an n-dimensional manifold and let N be p-dimensional, compact, orientable, and without boundary boundary. Let $\psi_0 : N \to M$ and $\psi_1 : N \to M$ be smooth and diffeotopic, i.e. there exists a smooth map $F : [0,1] \times N \to M$ such that

$$\psi_0 = F \circ \iota_0 \quad \text{and} \quad \psi_1 = F \circ \iota_1 ,$$

where ι_0 and ι_1 are the injection of N into $\{0\} \times N$ resp. $\{1\} \times N$. Show that for any closed p-form $\omega \in \Lambda_p(M)$ it holds that

$$\int_{N_0} \omega = \int_{N_1} \omega \,,$$

where

$$\int_{N_j} \omega := \int_N \psi_j^* \omega \,.$$

Discuss how the statement and its proof need to be modified for a manifold N with boundary.

Hint: The statement is, by definition, equivalent to $\int_N \psi_0^* \omega = \int_N \psi_1^* \omega$. In order to prove this statement, consider the form $F^* \omega \in \Lambda_p([0,1] \times N)$ and apply to it the homotopy operator $\mathrm{d} \circ K + K \circ \mathrm{d}$ from problem 28. Conclude that the form $(\psi_0^* - \psi_1^*)\omega$ is exact and use Stoke's theorem.

Problem 33: Closed forms and holomorphic functions

Let $U_\mathbb{C} \subset \mathbb{C}$ be open, $f : U_\mathbb{C} \to \mathbb{C}$ holomorphic, and $\gamma_\mathbb{C} : [0,1] \to U_\mathbb{C}$ a smooth curve. Discuss that the expression $\omega = f \, \mathrm{d}z$ defines a (complex valued) 1-form on the two-dimensional real manifold $U_{\mathbb{R}^2} = \{(x,y) \,|\, x + \mathrm{i}y \in U_\mathbb{C}\}$ such that for $\gamma_{\mathbb{R}^2} : [0,1] \to U_{\mathbb{R}^2}$, $\gamma_{\mathbb{R}^2}(t) = (\mathrm{Re}\, \gamma_\mathbb{C}(t), \mathrm{Im}\, \gamma_\mathbb{C}(t))$

$$\int_{\gamma_\mathbb{C}} f \, \mathrm{d}z = \int_{\gamma_{\mathbb{R}^2}} \omega \,.$$

(Note that the left hand side denotes the usual line integral in the complex plane.) Write ω in the basis representation with respect to the basis 1-forms $\mathrm{d}x$ and $\mathrm{d}y$ and show that ω is closed.

Problem 34: Cauchy's integral theorem

Show that the following version of Cauchy's integral theorem is a special case of the statement of problem 32.

Cauchy's integral theorem: Let $U \subset \mathbb{C}$ be open and let γ_1 and γ_2 be closed smooth curves in U that are diffeotopic as closed curves. Then it holds for every holomorphic function $f : U \to \mathbb{C}$ that

$$\int_{\gamma_1} f(z) \, \mathrm{d}z = \int_{\gamma_2} f(z) \, \mathrm{d}z \,.$$

Problem 35: The hairy ball theorem

Show that "one can't comb a hairy ball flat without creating a cowlick", which is a common paraphrase of the following precise statement:

On the n-dimensional sphere $S^n = \{x \in \mathbb{R}^{n+1} \,|\, \|x\| = 1\}$ with even dimension n every smooth vector field $X \in \mathcal{T}_0^1(S^n)$ has at least one zero.

Proceed by contradiction: Assume that there exists a vector field X without zero, which can be normalized to $\|X\|_{\mathbb{R}^{n+1}} = 1$ without loss of generality. Now use the property that $\langle X(x), x \rangle_{\mathbb{R}^{n+1}} = 0$ in order to construct a diffeotopy $F : [0,1] \times S^n \to S^n$ of $\psi_0 : S^n \to S^n \subset \mathbb{R}^{n+1}$, $\psi_0(x) = x$ and $\psi_1 : S^n \to S^n \subset \mathbb{R}^{n+1}$, $\psi_1(x) = -x$. Next find a nowhere vanishing volume form ω on S^n, e.g. by using the normal vector field $n(x) = x$ to S^n and the canonical volume form ε on $\mathbb{R}^{n+1}$. Finally show that for even n the map ψ_1 changes the orientation, i.e.

$$\int_{S^n} \omega = \int_{S^n} \psi_0^* \omega = -\int_{S^n} \psi_1^* \omega \,,$$

and derive a contradiction from there.