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Geometry in Physics

Homework Assignment # 9

Problem 36: Brouwer fixed-point theorem (smooth version)

Let $B^n := \{x \in \mathbb{R}^n \,|\, \|x\| \leq 1\}$ be the unit ball in $\mathbb{R}^n$ and $S^{n-1} = \partial B^n$ its boundary, the $(n-1)$-sphere.

Let $f : B^n \to B^n$ be a smooth map. Show that f has at least one fixed-point, i.e. that there exists at least one $x_0 \in B^n$ with $x_0 = f(x_0)$.

Guidance: Assume that no such fixed-point exists. Then every x defines an open half-line E_x starting in $f(x)$ and going through x, $E_x := \{f(x) + t(x - f(x)) \,|\, t \in (0, \infty)\}$. Let $F : B^n \to S^{n-1}$ be the function that maps x to the unique intersection of E_x with S^{n-1}. Show that F acts on $S^{n-1} \subset B^n$ as the identity, i.e. $F \circ \iota = \mathrm{id}$, where $\iota : S^{n-1} \to B^n$ is the natural injection. Apply Stoke's theorem to $\omega := F^*\varepsilon$ in order to construct a contradiction.

Remark: Brouwer's fixed-point theorem is the generalisation of this result to continuous functions f. One can obtain it from the smooth version proved here by applying the Stone-Weierstrass approximation theorem.

Problem 37: Explicit flows

Let $M := \{x \in \mathbb{R}^2 \,|\, |x| < 1\}$. Specify explicitly vector fields $Z \in \mathcal{T}_0^1(M)$, the associated maximal flows $\Phi^Z : D \to M$, and the domain D, such that:

(i) Z is complete,
(ii) Z is not complete.

Problem 38: Linearisation of vector fields at fixed-points

Let $X \in \mathcal{T}_0^1(M)$ be a smooth vector field and $x_0 \in M$ a zero of X, i.e. $X(x_0) = (x_0, 0)$. In a chart φ with $\varphi(x_0) = 0$ let

$$X_\varphi(q) = DX_\varphi(0)\, q + \mathcal{O}(\|q\|^2)$$

be the Taylor approximation of $X_\varphi := I \circ \varphi_* X$ at 0. Here Df denotes the usual differential of a function $f : \mathbb{R}^n \to \mathbb{R}^n$, i.e. the Jacobi matrix $Df(q)_{ij} = \partial_{q_j} f_i(q)$. Let ψ be a further chart with $\psi(x_0) = 0$ and $\Phi = \psi \circ \varphi^{-1}$ the transition map.

Show that

$$DX_\psi(0) = D\Phi(0)\, DX_\varphi(0)\, D\Phi(0)^{-1}$$

and conclude that the eigenvalues of the "linearisation" $DX_\varphi(0)$ and their multiplicities do not depend on the chosen chart.

Problem 39: Fixed-points for gradient fileds and Hamiltonian vector fields

Let $M = \mathbb{R}^2$ with coordinates $(q, p) \in \mathbb{R}^2$ and $H \in C^\infty(\mathbb{R}^2)$. Discuss the possible local behavior near fixed-points of the following two types of vector-fields by considering the linear approximation. Sketch in all cases the vector field and the local flow.

(i) Gradient fields: $$X_G(q,p) := \begin{pmatrix} \partial_q H(q,p) \\ \partial_p H(q,p) \end{pmatrix}$$

(ii) Hamiltonian vector fields: $$X_H(q,p) := \begin{pmatrix} \partial_p H(q,p) \\ -\partial_q H(q,p) \end{pmatrix}.$$