EBERHARD KARLS UNIVERSITÄT TÜBINGEN

Preparation course for the new Master Mathematical Physics Students 2018

1st October - 5th October, 2018

Metric and Topological Spaces

Exercise 1.
Let X be a set. Prove that

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

is a distance over X. In general we call d discrete distance and (X, d) discrete metric space.

Exercise 2.
Let (X, d) and (Y, d') be metric spaces. Let

$$L(X,Y) := \{f : X \to y \,|\, f \text{ is bounded}\},$$

$$d_\infty(f,g) := \sup_{x \in X} d'(f(x), g(x)).$$

Prove that $(L(X,Y), d_\infty)$ is a metric space.

Exercise 3.
We define

$$\ell^p := \left\{ \{x_n\} \subset \mathbb{R} \,|\, \sum_{n=1}^{+\infty} |x_n|^p < \infty \right\}, \qquad d_p(\{x_n\}, \{y_n\}) := \left(\sum_{n=1}^{+\infty} |x_n - y_n|^p \right)^{\frac{1}{p}}$$

Prove that (ℓ^p, d_p) is a complete metric space.

Exercise 4.
We define

$$C^0([-1,1], \mathbb{R}) := \left\{ f : [-1,1] \to \mathbb{R} \,,\ f \text{ continuous} \right\}, \qquad d_1(f,g) := \int_{-1}^{1} |f(x) - g(x)| \, dx$$

Prove that $\left(C^0([-1,1], \mathbb{R}), d_1\right)$ is a metric space. Prove also that it is not complete.

Exercise 5.
Let (X, d) be a complete metric space. Let $C \subset X$ be a closed set. Prove that (C, d) is complete.

Exercise 6.
We define

$$C^1([a,b], \mathbb{R}) := \left\{ f : [a,b] \to \mathbb{R} \,,\ f \text{ continuous with continuous derivative} \right\},$$

$$\overline{d}_{\infty,1}(f,g) := \sup_{x\in[a,b]} |f'(x) - g'(x)| \qquad d_{\infty,1}(f,g) := \overline{d}_{\infty,1}(f,g) + d_\infty(f,g).$$

Is $(C^1([a,b],\mathbb{R}), \overline{d}_{\infty,1})$ a metric space? Is $(C^1([a,b],\mathbb{R}), d_{\infty,1})$ a metric space?

Exercise 7.
Prove that if (X,d) is a metric space, then

$$\overline{d}(x,y) := \frac{d(x,y)}{1+d(x,y)}$$

is a distance on X. Prove also that d and $\overline{d}$ are topologically equivalent.

Exercise 8.
Let (X,d) be a metric space. Verify that $\varepsilon : X\times X \to \mathbb{R}$ such that

$$\varepsilon(x,y) := \min\{1, d(x,y)\} \quad \forall\, x,y\in X,$$

is a distance over X.

Exercise 9.

(a) Let $\{\tau_\alpha\}_{\alpha\in I}$ be a family of topologies on X, verify that $\cap_{\alpha\in I}\tau_\alpha$ is a topology on X.

(b) Give an example of two topologies τ_1 and τ_2 on a set X such that $\tau_1\cup\tau_2$ is not a topology on X.

Exercise 10.
Let X be a non empty set. Let τ and τ' be two topologies on X. Prove that the map id : $(X,\tau)\to(X,\tau')$ is continuous if and only $\tau' < \tau$.

Exercise 11.
Find a topological space (X,τ) (with τ different from the trivial or the discrete topology) such that every open set is also close. If in a topological space every open set is also closed, then it is true that every closed set is also open?

Exercise 12.
Let (X,d) be a discrete metric space. Determine the set $\mathcal{A}$ of its open set and for every $x\in X$ determine the set $\mathcal{D}(x)$ of the balls centered in x.

Exercise 13.
Prove that every metrizable and finite space is a discrete metric space.

Exercise 14.
Let (X,τ) be a separable topological space. Let τ' be a topology on X such that $\tau' < \tau$. Prove that (X,τ') is separable.

Exercise 15.
Let $S := \big\{\emptyset, \mathbb{R}, \{(-\infty,a],\, a\in\mathbb{R}\}\big\}$. Verify that S is not a topology on $\mathbb{R}$. Determine the topology generated by S, $\tau(S)$. Compare $\tau(S)$ with the following topology τ' such that

$$\tau' := \big\{\emptyset, \mathbb{R}, \{(-\infty,b),\, b\in\mathbb{R}\}\big\}$$