EBERHARD KARLS UNIVERSITÄT TÜBINGEN

Preparation course for the new Master Mathematical Physics Students 2018

1st October - 5th October, 2018

Measure Theory

Exercise 1.
Let X be a set, the symmetry difference of $A, B \in \mathcal{P}(X)$, denoted by $A\Delta B$ and defined by $A\Delta B := (A \setminus B) \cup (B \setminus A)$. Let $(X, \mathcal{A}, \mu)$ be a measure space. Show that

$$\mu(A\Delta B) \leq \mu(A\Delta C) + \mu(C\Delta B), \qquad \forall A, B, C \in \mathcal{A}.$$

Exercise 2.
Let $\mathcal{G}$ the family of open set in a topological space. Give an example in which it is a $\sigma-$algebra and one in which it is not.

Exercise 3.
Let $\mathcal{C} = \big\{[0,1], [2,\infty)\big\} \subset \mathcal{P}(\mathbb{R})$. Find the minimal $\sigma-$algebra generated $\sigma_0(\mathcal{C})$ such that $\mathcal{C} \subset \sigma_0(\mathcal{C})$.

Exercise 4.
Exhibit two topologies on $\mathbb{R}$ such that the corresponding family of of Borel set are different.

Exercise 5.
Let $f, g : \mathbb{R} \to \mathbb{R}$ continuous and equal a.e. in $\mathbb{R}$ (i.e. up to a null set) with respect to the Lebesgue measure. Prove that $f = g$ in $\mathbb{R}$.

Exercise 6.
For every $E \subset \mathbb{R}$, we define

$$m^*(E) := \Lambda^*([0,1] \cap E),$$

where Λ^* is the external Lebesgue measure on $\mathbb{R}$. Verify that m^* is an external measure and characterize the $\sigma-$algebra of the m^*-measureble sets. The measure obtained by restriction of m^* to such $\sigma-$algebra is translation invariant?

Exercise 7.
Let $\mu^* : \mathcal{P}(X) \to [0,\infty)$ defined by

$$\mu^*(\emptyset) := 0, \quad \mu^*(E) := 1 \ \forall E \subset X.$$

Verify that μ^* is an external measure on X and determine the $\sigma-$algebra of the μ^*-mesureable sets.

Exercise 8.
Let X a set, $f : X \to [0,\infty)$. Let $\mu^* : \mathcal{P}(X) \to [0,\infty]$ such that

$$\mu^*(E) := \begin{cases} 0 & \text{if } E = \emptyset, \\ \sup_{x\in E} f(x) & \text{otherwise.} \end{cases}$$

Then μ^* in an external measure?

Exercise 9.
Let X be a set, $x_0 \in X$. We define $\mathcal{A} := \{\emptyset, X, \{x_0\}, \{x_0\}^c\}$. Consider $\mu : \mathcal{A} \to [0, \infty)$ such that

$$\mu(\emptyset) = 0, \quad \mu(\{x_0\}) = \mu(X) = 1, \quad \mu(\{x_0\}^c) = 0.$$

Then $(X, \mathcal{A}, \mu)$ is a complete measure space? Answer to the same question if

$$\mu(\emptyset) = 0, \quad \mu(\{x_0\}) = 1, \quad \mu(X) = 3, \quad \mu(\{x_0\}^c) = 2.$$

Exercise 10.
Let $f : [a, b] \to \mathbb{R}$ continuous. Prove that the graph of f, i.e.,

$$\mathcal{G}_f := \{(x, f(x)), x \in [a, b]\}$$

is a measureable subset of $\mathbb{R}^2$ with vanishing Lebesgue measure.

Exercise 11.
Let $(X, \mathcal{A}, \mu)$ be a measure space. Show that for any $A, B \in \mathcal{A}$, we have

$$\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B).$$

Exercise 12.
Let $E \subset \mathbb{R}$. Prove that the following statements are equivalent:

1. E is Lebesgue mesureable
2. $\forall \varepsilon > 0$, there exists an open set O such that $E \subset O$ and $\Lambda^*(O \setminus E) \leq \varepsilon$.
3. There exists a set G that one can write as the intersection of countable many open sets such that $E \subset G$ and $\Lambda^*(G \setminus E) = 0$.

Exercise 13.
Let $E \in \mathcal{L}$ (where $\mathcal{L}$ is the Lebesgue $\sigma-$algebra) such that $\Lambda(E) < \infty$. Define a real-valued function φ_E on $\mathbb{R}$ by setting

$$\varphi_E(x) := \Lambda(E \cap [-\infty, x]), \quad \text{for } x \in \mathbb{R}.$$

Show that φ_E is an increasing function on $\mathbb{R}$. Show that φ_E satisfies the Lipschitz condition on $\mathbb{R}$, that is

$$|\varphi_E(x') - \varphi_E(x'')| \leq |x' - x''|, \quad \forall x', x'' \in \mathbb{R}$$

Exercise 14.
Let $E \in \mathcal{L}$ such that $\Lambda(E) = 1$. Show that there exists a Lebesgue mesureable set $A \subset E$ such that $\Lambda(A) = \frac{1}{2}$.

Exercise 15.

- Let $E \subset \mathbb{R}$. Show that $\mathcal{F} = \{\emptyset, E, E^c, \mathbb{R}\}$ is the $\sigma-$algebra of subset of $\mathbb{R}$ generated by $\{E\}$.
- If S and τ are collections of subsets of $\mathbb{R}$, then it is true that

$$\sigma(S \cup \tau) = \sigma(S) \cup \sigma(\tau)\,?$$

Exercise 16. Let $\mathbb{Q}$ be the set of the rational numbers in $\mathbb{R}$.

1. Show that $\mathbb{Q}$ is a null set in $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \Lambda)$.
2. Show that $\mathbb{Q}$ is a F_σ-set
3. Show that there exists a $G_\delta-$set such that $\mathbb{Q} \subset G$ and $\Lambda(G) = 0$.
4. Show that the set of all irrational numbers in $\mathbb{R}$ is a $G_\delta-$set.

Where we denote by $F_\sigma-$set a set given by the union of countable many closed sets and by $G_\delta-$set a set given by the intersection of countable many open sets.

Exercise 17. Let $E \in \mathcal{L}$, with $\Lambda(E) > 0$. Prove that for every $\alpha \in (0, 1)$, there exists a finite open interval I such that

$$\alpha\Lambda(I) \leq \Lambda(E \cap I) \leq \Lambda(I).$$

Exercise 18. Prove that for everty $E \subset \mathbb{R}$, there exists a $G_\delta-$set G, with $G \subset \mathbb{R}$, such that $E \subset G$ and $\Lambda^*(G) = \Lambda^*(E)$.