EBERHARD KARLS
UNIVERSITÄT
TÜBINGEN

Preparation course for the new Master Mathematical Physics Students 2018

1st October - 5th October, 2018

Measurable Functions

Exercise 1.
Let $(X, \mathcal{A})$, $(X', \mathcal{A}')$, $(X'', \mathcal{A}'')$ be measurable spaces. Let $f : X \to X'$ be $(\mathcal{A}, \mathcal{A}')-$measurable and $g : X' \to X''$ $(\mathcal{A}', \mathcal{A}'')-$measurable. Prove that $g \circ f : X \to X''$ is $(\mathcal{A}, \mathcal{A}'')-$measurable.

Exercise 2. Let $(X, \mathcal{A})$ be a measurable space. Let $F \subset X$. Prove that the inclusion $j_F : F \to X$ is $(\mathcal{A} \cap F, \mathcal{A})-$measurable.

Exercise 3. Let $(X, \mathcal{A})$, $(X, \mathcal{A}')$ be two measurable spaces. Let $f : X \to X'$. Let $\{F_n\} \subset \mathcal{A}$ such that $X = \cup_{n=1}^{\infty} F_k$. Prove that the following statements are equivalent:

1. f is $(\mathcal{A}, \mathcal{A}')-$measurable
2. $\forall k \in \mathbb{N}$, $f\big|_{F_k} : F_l \to X'$ is $(\mathcal{A} \cap F, \mathcal{A}')-$measurable.

Exercise 4. Consider $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, let $c > 0$ arbitrary. For any $E \in \mathcal{B}(\mathbb{R})$,

$$\mu(E) := \begin{cases} \sum_{k \in E \cap \mathbb{N}} c^k & \text{if } E \cap \mathbb{N} \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

Verify that μ is a measure, is μ a $\sigma-$finite measure? Is μ absolutely continuous with respect to the Lebesgue measure? Is the Lebesgue measure absolutely continuous with respect to μ?

Exercise 5. Let $(X, \mathcal{A}, \mu)$ be a measure space. If $s \in S_+(X, \mathcal{A})$, define

$$\varphi(E) := \int_E s \, d\mu, \quad \forall E \in \mathcal{A}.$$

Prove that $\varphi : \mathcal{A} \to [0, \infty]$ is a measure on X.

Exercise 6. The function

$$f(x) := \begin{cases} \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is measurable in $(\mathbb{B}, \mathcal{L}, \Lambda)$?

Exercise 7. Let $f : \mathbb{R} \to \mathbb{R}$ be a derivable function. Is the derivative o f, $f' : \mathbb{R} \to \mathbb{R}$ a measurable function?

Exercise 8. Consider

$$f_n(x) := n \exp^{-nx} \chi_{(0,1]}(x), \quad g_n(x) := n \exp^{-nx} \chi_{[0,1]}(x)$$

Are the two sequences point wise convergent? Are the two sequences a.e. convergent (w.r.t. the Lebesgue measure)?

Exercise 9. Prove that if f is a function such that $f^2 : \mathbb{R} \to [0, \infty)$ is measurable, this does not imply that f is measurable.

Exercise 10. Let f be a function with measurable domain D. Show that f is measurable if and only if the function g such that

$$g(x) := \begin{cases} f(x) & x \in D \\ 0 & x \neq D \end{cases}$$

is measurable.

Exercise 11.

Let f be a real-valued increasing function on $\mathbb{R}$. Show that if it is Borel measurable, then it is Lebesgue measurable.

Exercise 12. Let f be measurable and B a Borel set. Prove that $f^{-1}(B)$ is a measurable set.

Exercise 13. Prove that

1. If $g : \mathbb{R} \to \mathbb{R}$ is continuous and $f : \mathbb{R} \to \mathbb{R}$ is measurable. Then $g \circ f$ is measurable.
2. If f is measurable, then $|f|$ is measurable. Does the converse hold?

Exercise 14. Show that

$$\chi_{a \cap B} = \chi_A \cdot \chi_B, \quad \chi_{A \cup B} = \chi_A + \chi_B - \chi_A \cdot \chi_B, \quad \chi_{A^c} = 1 - \chi_A$$

Exercise 15. If $\{f_n\}$ is a sequence of measurable function on $D \subset \mathbb{R}$. Prove that

$$\{x \in D \,\text{s.t.} \lim_{n \to +\infty} f_n(x) \text{ exists } \}$$

is a measurable set.

Exercise 16. Prove that $f : \mathbb{R} \to [0, \infty)$ such that

$$f(x) := \begin{cases} 1 & x \leq 0 \\ x^2 & x > 0 \end{cases}$$

is a measurable function.

Exercise 17. Let $(X, \mathcal{A}, \mu)$ be a measure space Let f be extended real-valued, $\mathcal{A}-$measurable, integrable function on X. Let

$$E_n := \{x \in X \,\text{s.t.}\, |f(x)| \geq n\}, \quad \forall n \in \mathbb{N}$$

Show that $\lim_{n \to +\infty} \mu(E_n) = 0$.

Exercise 18. Let f be a real-valued uniformly continuous function on $[0, \infty)$. Show that if f is Lebesgue integrable on $[0, \infty]$, then $\lim_{x \to \infty} f(x) = 0$.