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Preparation course for the new Master Mathematical Physics Students 2018

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Integrable Functions

Exercise 1.
Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing monotone function. Prove that f is Lebesgue measurable.

Exercise 2.
Let $(X, \mathcal{A}, \mu)$ be a measure space, with $\mu(X) < \infty$. Let $f \in L^\infty(X, \mathcal{A}, \mu)$. Prove that

$$\|f\|_\infty = \lim_{p \to +\infty} \|f\|_p$$

Exercise 3.
Consider

$$f_n : [0,1] \to \mathbb{R},\ f_n(x) := 1 - e^{-\frac{x^2}{n}}, \qquad g_n : (0,1] \to \mathbb{R},\ g_n(x) := \frac{1 - e^{-\frac{x^2}{n}}}{\sqrt{x}}$$

Study the pointwise convergence of $\{f_n\}$ and $\{g_n\}$ and compute $\lim_{n\to+\infty} \int_{[0,1]} f_n, d\Lambda$ and $\lim_{n\to+\infty} \int_{[0,1]} g_n, d\Lambda$.

Exercise 4.
Let $f : [1, \infty) \to \infty$, $f(x) := 1/[x]$ ($[x]$ = integer part of x). Consider $f_n : [1, \infty) \to \mathbb{R}$, $f_n(x) := f(x)\chi_{[1,n)}(x)$ for all $n \geq 2$. Prove that f_n is integrable in $[1, \infty)$ and calculate the integrale. Prove also that f is not integrable in $[1, \infty)$.

Exercise 5.

1. Consider $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu^\#)$. We define $g(n) := 1/n!$, evaluate $\int_{\mathbb{N}} g\, d\mu^\#$.
2. Let $f_\alpha(n) := \frac{1}{n^\alpha}$, $\alpha \in \mathbb{R}$, find the value of α such that $f_\alpha \in L^1(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu^\#)$.

Exercise 6.
Let $(X, \mathcal{A}, \mu)$ be a measure space, with $\mu(X) < \infty$. Let $f \in \mathcal{X}, \mathcal{A}$. Let

$$E_n := \{x \in X \,|\, n - 1 < |f(x)| \leq n\}, \quad n \in \mathbb{N}.$$

Prove that

$$f \in L^1(X, \mathcal{A}, \mu) \Leftrightarrow \sum_{n=1}^{\infty} n\, \mu(E_n) < \infty$$

Is this still true if $\mu(X) = \infty$?

Exercise 7.
Let $f \in L^1(X, \mathcal{A}, \mu)$, $f \geq 0$ a.e. We define

$$f_n := \frac{nf}{n + f},$$

Prove that $\{f_n\} \subset L^1(X, \mathcal{A}, \mu)$ and that $\|f_n - f\|_{L^1} \to 0$ as $n \to \infty$

Exercise 8.
Let $f \in L^1(\mathbb{R}, \mathcal{L}, \Lambda)$ such that $\int_{\mathbb{R}} |x||f(x)|\, d\Lambda < \infty$. Prove that $F : [0,1] \to [0, \infty]$, such that

$$F(\alpha) := \int_{\mathbb{R}} |x|^{\alpha}|f(x)|\, d\Lambda,$$

is a continuous function.

Exercise 9.
Let $(X, \mathcal{A}, \mu)$ be a measure space. Show that if $f_n \to f$ in measure and $g_n \to g$ in measure, then $f_n + g_n \to f + g$ in measure.

Exercise 10.
Let $(X, \mathcal{A}, \mu)$ be a finite measure space. LetΦ the set of the function in $\mathcal{M}(X, \mathcal{A})$ where we identify functions that are equal a.e. on X. Prove that

$$\rho(f, g) := \int_X \frac{|f-g|}{1+|f-g|}, \quad \text{for } f, g \in \Phi$$

is a distance on X.

Exercise 11.

1. Let $(\mathbb{R}, \mathcal{L}, \Lambda)$ be a measurable space. For every $f : \mathbb{R} \to \mathbb{R}$, define $f_a(x) := f(x-a)$. Prove that
$$\int_{\mathbb{R}} f\, d\Lambda = \int_{\mathbb{R}} f_a\, d\Lambda$$
2. Let f be a non-negative, real-valued Lebesgue measurable function on $\mathbb{R}$. Show that if $\sum_{n=1}^{\infty} f(x+n)$ is Lebesgue integrable on $\mathbb{R}$, then $f = 0$ a.e. on $\mathbb{R}$.

Exercise 12.
Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $\{f_n\}_{n\in\mathbb{N}} \in L^1(X, \mathcal{A}, \mu)$. Suppose that $\|f_n - f\|_{L^1} \to 0$ as $n \to \infty$, show that

- $f_n \to f$ in measure on X.
- $\lim_{n\to+\infty} \int_X |f_n| = \int_X |f|$

Exercise 13.
Let $(X, \mathcal{A}, \mu)$ be a measure space.

1. Let f be a real-valued measurable function on X such that $f \in L^1(X) \cap L^{\infty}(X)$. Show that $f \in L^p(X)$ for every $p \in [1, \infty]$.
2. Let $\mu(X) < \infty$. Let $p, q \in [1, \infty]$ such that $p \leq q$. Prove that $L^q(X, \mathcal{A}, \mu) \subset L^p(X, \mathcal{A}, \mu)$.

Exercise 14.
Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $\{f_n\} \subset L^1(X, \mathcal{A}, \mu)$ and $f \in L^1(X, \mathcal{A}, \mu)$ such that

- $\lim_{n\to\infty} f_n = f$ a.e. on X,
- $\lim_{n\to\infty} \int_X |f_n|\, d\mu = \int_X |f|\, d\mu$.

Prove that

$$\lim_{n\to\infty} \int_X |f_n - f| = 0.$$