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Preparation course for the new Master Mathematical Physics Students 2018

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Second exercise sheet

1. Verify that the following spaces are pre-Hilbert spaces[1].

   a) Let $\Omega \subseteq \mathbb{R}$ be a Lebesgue-measurable set. Consider $L^2(\Omega)$ equipped with the map $\langle \cdot , \cdot \rangle_{L^2(\Omega)} \colon L^2(\Omega) \times L^2(\Omega) \to \mathbb{C}$ defined as

   $$\langle u, v\rangle_{L^2(\Omega)} := \int_\Omega \mathrm{d}x \, \overline{u(x)} v(x) \quad \forall u, v \in L^2(\Omega).$$

   b) Consider $\ell^2(\mathbb{Z})$ equipped with the map $\langle \cdot , \cdot \rangle_{\ell^2(\mathbb{Z})} \colon \ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z}) \to \mathbb{C}$ defined as

   $$\langle u, v\rangle_{\ell^2(\mathbb{Z})} := \sum_{n\in\mathbb{Z}} \overline{u_n} v_n \quad \forall u, v \in \ell^2(\mathbb{Z}).$$

   c) Consider $L^2(\mathbb{T}^1)$ where $\mathbb{T}^1 := \mathbb{R}/(2\pi\mathbb{Z})$ equipped with the map $\langle \cdot , \cdot \rangle_{L^2(\mathbb{T}^1)} \colon L^2(\mathbb{T}^1) \times L^2(\mathbb{T}^1) \to \mathbb{C}$ defined as

   $$\langle u, v\rangle_{L^2(\mathbb{T}^1)} := \int_0^{2\pi} \mathrm{d}x \, \overline{u(x)} v(x) \quad \forall u, v \in L^2(\mathbb{T}^1).$$

2. Consider the densely defined operator $(\mathcal{D}(\Delta_{\mathrm{per}}), \Delta_{\mathrm{per}})$ where

   $$\mathcal{D}(\Delta_{\mathrm{per}}) := \{f \in C^2(\mathbb{R}) \text{ such that } f \text{ is } 2\pi\mathbb{Z}\text{-periodic}\}$$

   and

   $$\Delta_{\mathrm{per}} \colon \mathcal{D}(\Delta_{\mathrm{per}}) \subseteq L^2(\mathbb{T}^1) \to L^2(\mathbb{T}^1) \text{ such that } \Delta_{\mathrm{per}} u := \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \quad \forall u \in \mathcal{D}(\Delta_{\mathrm{per}}).$$

   Verify that $(\mathcal{D}(\Delta_{\mathrm{per}}), \Delta_{\mathrm{per}})$ is symmetric.

[1] Remember to check that the following definitions of inner products make sense.