EBERHARD KARLS
UNIVERSITÄT
TÜBINGEN

Preparation course for the new Master Mathematical Physics Students 2018

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Third exercise sheet

1. Consider the Cauchy problem associated to the *heat equation* on the 1-dimensional torus $\mathbb{T}^1$:
$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) & x \in \mathbb{T}^1,\, t > 0 \\ u(0,x) = u_0(x), u_0 \in L^2(\mathbb{T}^1) \text{ (fixed).} \end{cases}$$
Find the solution $u(t,\,\cdot\,) \in L^2(\mathbb{T}^1)$ for every $t \geq 0$ as a *Fourier series* and study the *asymptotic behavior*, namely $L^2(\mathbb{T}^1)\text{-}\lim_{t\to\infty} u(t,\,\cdot\,)$, interpreting this result physically.

2. Let $\gamma \in \mathbb{R}$, we define
$$(T_\gamma f)(x) := f(x-\gamma) \quad \forall f \in L^2(\mathbb{T}^1)$$
$$(\rho_\gamma a)_n := \mathrm{e}^{-\mathrm{i}\gamma n} a_n \quad \forall a \in \ell^2(\mathbb{Z}).$$
Verify that both the operators T_γ and ρ_γ are unitary for every $\gamma \in \mathbb{R}$.

3. Let $\mathcal{F}\colon L^2(\mathbb{T}^1) \to \ell^2(\mathbb{Z})$ be the Fourier transform. Prove that the following properties are satisfied:

   a)
   $$\mathcal{F}\,T_\gamma = \rho_\gamma\,\mathcal{F} \quad \forall \gamma \in \mathbb{R}.$$

   b) If $f \in C^1(\mathbb{T}^1)$, then
   $$\left(\mathcal{F}\frac{\mathrm{d}}{\mathrm{d}x}f\right)_n = \mathrm{i}n(\mathcal{F}f)_n \quad \forall n \in \mathbb{Z}.$$