EBERHARD KARLS UNIVERSITÄT TÜBINGEN

Preparation course for the new Master Mathematical Physics Students 2018

8th October - 12th October, 2018

Fourth exercise sheet

1. Find the solution $x(\cdot)$ to the Newton's equations of
   a) the motion in $\mathbb{R}^3$ for a point particle with mass $m > 0$ under the *gravitational force* $F_g := (0, 0, -mg)$ with $g > 0$, assuming that $x(t_0) = x_0 \in \mathbb{R}^3$ and $\dot{x}(t_0) = v_0 \in \mathbb{R}^3$, with $t_0 \in \mathbb{R}$;
   b) the motion in $\mathbb{R}$ for a point particle with mass $m > 0$ subject to the *Hooke*'s force $F_H(x) := -kx$ with $k > 0$, assuming that $x(t_0) = x_0 \in \mathbb{R}$ and $\dot{x}(t_0) = v_0 \in \mathbb{R}$, with $t_0 \in \mathbb{R}$.
2. Write the Lagrangians for the two systems in the previous exercise.
3. Let D be an open set of $\mathbb{R}^{3N}$, $N \geq 1$. Consider the Lagrangian $\mathcal{L}\colon D \times \mathbb{R}^{3N} \to \mathbb{R}$ defined as
$$\mathcal{L}(q, \xi) := \sum_{k=1}^{N} \frac{1}{2} m_k \left|\xi_k\right|^2 - U(q),$$
   where $m_k > 0$ for all $k \in \{1, \dots, N\}$ and $U \in C^n(D, \mathbb{R})$, $n \geq 2$. Given a *coordinate transformation in the configuaration space* $\psi\colon \Lambda \times \mathbb{R} \to D$ such that $q = \psi(Q, t)$, compute the Lagragian $\widetilde{\mathcal{L}}\colon \Lambda \times \mathbb{R}^{3N} \to \mathbb{R}$ in the new coordinates.
4. Consider the Lagrangian
$$\mathcal{L}(q, \xi) = \frac{1}{2} m \left|\xi\right|^2 - U(q), \quad (q, \xi) \in \mathbb{R}^2 \times \mathbb{R}^2,$$
   where $U(q) = U(|q|)$ (U is called a *central potential*).
   a) Which is a "suitable" coordinate transformation to rewrite the Lagrangian $\mathcal{L}$? Once you have found it, write $\mathcal{L}$ in the new coordinates.
   b) Using the invariance of the Euler–Lagrangian equations under *coordinate transformations in the configuaration space*, study the motion in the plane taking advantage of a *cyclic variable*.
5. Let D be an open set of $\mathbb{R}^n$. Let $\mathcal{L} \in C^k(D \times \mathbb{R}^n \times \mathbb{R}, \mathbb{R})$ with $k \geq 2$ be a Lagrangian. We define the *generalized energy* as
$$\mathcal{H}(q, \xi, t) := \left\langle \xi, \frac{\partial \mathcal{L}}{\partial \xi} \right\rangle_{\mathbb{R}^n} - \mathcal{L}(q, \xi, t).$$
   Prove the following statement:

   If $\phi\colon [t_1, t_2] \subset \mathbb{R} \to D$ is a solution of the Euler–Lagrange equations, then
$$\frac{d}{dt} \mathcal{H}(\phi(t), \dot{\phi}(t), t) = -\frac{\partial \mathcal{L}}{\partial t}(\phi(t), \dot{\phi}(t), t) \quad \forall t \in [t_1, t_2].$$