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Preparation course for the new Master Mathematical Physics Students 2018

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First exercise sheet

1. Let $\mathcal{H}$ be a (N-dimensional) Hilbert space with inner product $\langle\,\cdot\,,\,\cdot\,\rangle$.

   a) Prove that the Cauchy–Schwarz inequality holds

   $$|\langle u,v\rangle| \le \|u\|\,\|v\| \quad \forall u,v\in\mathcal{H}.$$

   b) Defining

   $$\|u\| := \langle u,u\rangle^{1/2} \quad \forall u\in\mathcal{H},$$

   verify that $\|\cdot\|$ is a norm on $\mathcal{H}$.

2. Let X be a set and

   $$d(x,y) := \begin{cases} 1 & \text{if } x\neq y \\ 0 & \text{if } x=y. \end{cases}$$

   Verify that (X,d) is a metric space. The metric d is called the *discrete distance*.

3. Let $-\infty<a<b<\infty$. Consider

   $$C^0([a,b],\mathbb{R}) := \{f\colon [a,b]\to\mathbb{R} \text{ such that } f \text{ is continuous}\}.$$

   We define for any $f,g\in C^0([a,b],\mathbb{R})$

   $$d_\infty(f,g) := \sup_{x\in[a,b]} |f(x)-g(x)|\,, \qquad d_1(f,g) := \int_a^b \mathrm{d}x\,|f(x)-g(x)|$$

   and

   $$d_{1/2}(f,g) := \int_a^{\frac{a+b}{2}} \mathrm{d}x\,|f(x)-g(x)|\,.$$

   Verify that $\big(C^0([a,b],\mathbb{R}),d_\infty\big)$ and $\big(C^0([a,b],\mathbb{R}),d_1\big)$ are metric spaces, instead $\big(C^0([a,b],\mathbb{R}),d_{1/2}\big)$ is not a metric space.

4. Let $-\infty<a<b<\infty$. Consider

   $$C^1([a,b],\mathbb{R}) := \{f\colon [a,b]\to\mathbb{R} \text{ such that } f \text{ is continuous with continuous derivative } f'\}.$$

   We define for any $f,g\in C^1([a,b],\mathbb{R})$

   $$d_{\infty,1}(f,g) := d_\infty(f',g'),$$

   where $d_\infty(\,\cdot\,,\,\cdot\,)$ is introduced in the previous exercise. Is $\big(C^1([a,b],\mathbb{R}),d_{\infty,1}\big)$ a metric space? If not, define a distance $\tilde{d}$ on this vector space.

5. Let (X, d) be a metric space and $x_0 \in X$ fixed. Verify that the function $d_{x_0} \colon X \to \mathbb{R}$ such that $d_{x_0}(x) := d(x_0, x)$ is continuous.

6. Let $(X, \|\cdot\|)$ be a normed space. Verify that

$$d(x, y) := \|x - y\| \text{ is a metric on } X.$$

7. Let X a N-dimensional (complex) vector space. Prove that any two norms $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ are *equivalent* in the sense that

$$C_1 \|u\|_\alpha \leq \|u\|_\beta \leq C_2 \|u\|_\alpha \quad \forall u \in X,$$

where C_1, C_2 are independent of u.

8. Assume that $(X, \|\cdot\|_X)$ is a finite-dimensional (complex) vector space and $(Y, \|\cdot\|_Y)$ is a Banach space. Show that every linear operator $A \colon X \to Y$ must be bounded.