Mathematical Quantum Theory
Exercise sheet 10
01.02.2019

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Exercise 1. [14 points] Consider the self-adjoint operators A, B on $L^2([-1,1])$ where A is the multiplication by x and B is the multiplication by x^2. Prove that:

(i) $f : [-1,1] \to \mathbb{R}$, $f(x) = 1$ is a cyclic vector of A, that is:

$$L^2([-1,1]) = \overline{\operatorname{span}\{A^n f \mid n \in \mathbb{N}\}} \ .$$

(ii) The characteristic function $\chi_{[0,1]}$ is not a cyclic vector of A.

(iii) The operator B has no cyclic vectors. *[Hint. Suppose there exists a cyclic vector f. Find a function g that is orthogonal to $B^n f$ for all n.]*

Exercise 2. [13 points]

(i) Let A be a densely defined, symmetric operator on $\mathcal{H}$ such that $D(A)$ contains an orthonormal basis of $\mathcal{H}$ consisting of eigenvectors of A. Prove that A is essentially selfadjoint.

(ii) Let $\mathcal{H} = L^2([0,1])$, and consider the densely defined operator A_N given by:

$$D(A_N) = \{\psi \in C^2([0,1]) \mid \psi'(0) = 0 = \psi'(1)\}$$
$$A_N \psi = -\psi'' \ , \qquad \text{(Neumann Laplacian)}$$

Prove that A_N is symmetric, and essentially selfadjoint. *[Hint. Recall that $\{\sin(n\pi x)\}$ is a basis for $L^2([0,1])$].*

Exercise 3. [13 points] Let $T : L^2([0,1]) \to L^2([0,1])$ be given by:

$$(Vf)(x) := \int_0^x f(y)dy \ .$$

Prove the following:

(i) $T \in \mathcal{L}(L^2([0,1]))$.

(ii) T is compact.
[Hint. Recall that operators of the form $(Kf)(x) = \int_\Omega dy\, k(x,y) f(y)$ with $k(x,y) \in L^2(\Omega \times \Omega)$ are compact]

(iii) $\sigma_{\mathrm{p}}(T) = \emptyset$.

(iv) $\sigma_{\mathrm{r}}(T) = \emptyset$.

(v) $\sigma(T) = \{0\}$.

(vi) $T + T^*$ is an orthogonal projection with $\dim \operatorname{Ran}(T + T^*) = 1$.