Mathematical Quantum Theory
Exercise sheet 2
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Exercise 1. [10 points] Let $1 \le p < \infty$, and let $A = (0,1) \subset \mathbb{R}$. Moreover, let $I = (a,b) \subset A$. For each $\varepsilon > 0$, find a function $g_\varepsilon \in C_c(A)$ such that $g_\varepsilon \to \mathbb{1}_I$ in $L^p(\Omega)$, with:

$$\mathbb{1}_I(x) = \begin{cases} 1 & x \in I \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 2. [10 points] Let $\mathcal{H}$ be a Hilbert space, and let $(x_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ be a sequence. We say that x_n converges weakly to $x \in \mathcal{H}$, $x_n \overset{n\to\infty}{\rightharpoonup} x$, if and only if, for all $y \in \mathcal{H}$, $\langle y, x_n - x\rangle \to 0$ as $n \to \infty$. Prove that the following statements are equivalent.

(a) x_n converges to x in norm, that is $\|x_n - x\|_{\mathcal{H}} \to 0$ as $n \to \infty$.

(b) $x_n \overset{n\to\infty}{\rightharpoonup} x$ and $\lim_{n\to\infty} \|x_n\| = \|x\|$.

Hint. Recall the reverse triangle inequality:

$$|\|y\| - \|x\|| \le \|y - x\| \, .$$

Exercise 3. [20 points] Let $\eta \in C_c^\infty(\mathbb{R}^d)$ such that:

(i) $\eta \ge 0$ and $\eta = 0$ if $|x| \ge 1$.

(ii) $\int_{\mathbb{R}^d} dx\, \eta(x) = 1$.

For all $\varepsilon > 0$, let $\eta_\varepsilon(x) := \varepsilon^{-d}\eta(x/\varepsilon)$. The function η_ε is called a *mollifier.* We define the *mollification* of f the function:

$$(\eta_\varepsilon * f)(x) := \int_{\mathbb{R}^d} \eta_\varepsilon(x-y) f(y)\, dy \, .$$

(a) Let $f \in L^p(\mathbb{R}^d)$, $1 \le p < \infty$. Prove that $\|\eta_\varepsilon * f\|_{L^p} \le \|f\|_{L^p}$.
Hint. Use Hölder inequality. Split $\eta = \eta^{1/p}\eta^{1/p'}$ *with* $\frac{1}{p} + \frac{1}{p'} = 1$.

(b) Let $\Omega \subseteq \mathbb{R}^d$ open, and let $f \in L^p(\Omega)$. Define f to be zero outside Ω. Show that $\eta_\varepsilon * f \in C^\infty(\mathbb{R}^d)$.
Hint. Recall dominated convergence.

(c) Let $\Omega' \subset \Omega$, Ω, Ω' open. Let $f \in C_c(\Omega')$ and zero outside Ω'. Show that $\eta_\varepsilon * f \in C_c(\Omega)$, provided $\varepsilon > 0$ is small enough.

(d) Let $f \in C_c(\Omega)$ and zero outside Ω. Prove that $\eta_\varepsilon * f$ converges uniformly to f as $\varepsilon \to 0$. Conclude that $\eta_\varepsilon * f \to f$ in $L^p(\Omega)$ as $\varepsilon \to 0$.

(e) Let $f \in L^p(\Omega)$, Show that for every $\delta > 0$ there exists $g \in C_c^\infty(\Omega)$ such that $\|f - g\|_{L^p(\Omega)} \le \delta$.
Hint. Use that $C_c(\Omega)$ *is dense* $L^p(\Omega)$ *for* $1 \le p < \infty$.

(f) Let $f \in L^p(\Omega)$. Prove that if f is continued with zero outside Ω, then $\|\eta_\varepsilon * f - f\|_{L^p(\Omega)} \to 0$ as $\varepsilon \to 0$.