Mathematical Quantum Theory
Exercise sheet 3
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Exercise 1. [10 points] Let $\eta : \mathbb{R} \to \mathbb{R}$ be given by:

$$\eta(x) = \begin{cases} x & x \geq 0 \\ -x & x < 0. \end{cases} \tag{1}$$

(i) Prove that $T_\eta \in \mathcal{S}'(\mathbb{R})$.

(ii) Compute $\frac{d^n}{dx^n} T_\eta(f)$ for all $n \in \mathbb{N}$.

(iii) Prove that there exists no function $g_n \in C^\infty_{\text{pol}}(\mathbb{R}^d)$ such that $\frac{d^n}{dx^n} T_\eta(f) = T_{g_n}$, for all $n \in \mathbb{N}$.
[Hint. Use that the delta distribution cannot be represented by a function.]

Exercise 2. [10 points] Let $\mathcal{F} : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$. Prove that

$$\mathcal{F}(\partial_x^\alpha T) = (ix)^\alpha \mathcal{F}(T) \; . \tag{2}$$

Exercise 3. [10 points] Let $g \in \mathcal{S}(\mathbb{R}^d)$. Let $T \in \mathcal{S}'(\mathbb{R}^d)$, $g_y(x) := g(x-y)$, and $\xi(y) := T(g_y)$. Prove that $\xi \in C^\infty_{\text{pol}}(\mathbb{R}^d)$.

Exercise 4. [10 points] Prove the uniqueness of the solution of the free Schrödinger equation on $\mathcal{S}'(\mathbb{R}^d)$.

[Hint. Suppose that $\psi(t), \varphi(t)$ are two solutions in $C^\infty(\mathbb{R}_t, \mathcal{S}'(\mathbb{R}^d))$, with $\psi(0) = \varphi(0)$. Define $\xi(t) = \psi(t) - \varphi(t)$, and let:

$$\widetilde{\xi}(t) = \mathcal{F}^{-1} e^{i\frac{|k|^2}{2}t} \mathcal{F} \xi(t) \; . \tag{3}$$

Show that $\widetilde{\xi}(t) = 0$ for all times, and use this fact to prove that $\psi(t) = \varphi(t)$.]