Mathematical Quantum Theory
Exercise sheet 4
16.11.2018

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Exercise 1. [10 points] *i)* Consider the following linear operator, from $L^2([0,\infty))$ to $L^2([0,\infty))$:

$$(L\psi)(t) = \psi(2t) \ . \tag{1}$$

Check that L is continuous and compute the norm.

ii) Let B_1 be the unitary circle in $\mathbb{R}^2$, $\{x \in \mathbb{R}^2 \mid |x| \leq 1\}$. Consider the following linear operator from $L^2(B_1)$ to $L^2(B_1)$:

$$(L\psi)(x,y) = \psi(y,-x) \ . \tag{2}$$

Check that L is continuous and compute the norm.

Exercise 2. [10 points] Let (c_k) be a sequence in $\mathbb{C}$ such that $|c_k| \leq Ck^p$ for some $p \in \mathbb{R}_+$. Prove that the series:

$$\sum_{k=1}^{\infty} c_k \sin kx \tag{3}$$

converges in $S'(\mathbb{R}^d)$.

[That is, show that the sum converges if evaluated against any test function $f \in \mathcal{S}(\mathbb{R}^d)$.]

Exercise 3. [10 points] Let $a(x) \in C^{\infty}_{\mathrm{pol}}(\mathbb{R})$. Solve the following differential equation in $\mathcal{S}'(\mathbb{R})$:

$$\frac{d}{dx}T + a(x)T = \frac{d}{dx}\delta \ . \tag{4}$$

Exercise 4. [10 points] *i)* Prove that $L^p(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$ for all $1 \leq p \leq \infty$.

[Functions in L^p are associated to distributions via the identification $f \to T_f$, for $f \in L^p$.]

ii) Prove that $\lim_{k\to\infty} \|u_k - u\|_p = 0$ (*u_k converges to u in L^p*) implies that $T_{u_k} \overset{*}{\rightharpoonup} T_u$ in $\mathcal{S}'(\mathbb{R}^d)$.