# Mathematical Quantum Theory
# Exercise sheet 5
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**Exercise 1. [10 points]** Let $H \in \mathcal{L}(\mathcal{H})$, $H = H^*$. Prove that:

$$U(t) := e^{-iHt} = \sum_{n=0}^{\infty} \frac{(-iHt)^n}{n!} \tag{1}$$

is a strongly continuous one-parameter group of unitary operators, with generator $H$. Moreover, prove that $U(t)$ is norm differentiable.

**Exercise 2. [10 points]** Let $\mathcal{D}'(\mathbb{R})$ be the dual space of $C_c^\infty(\mathbb{R})$ (*space of distributions*). Let $\{x_k\} \subset \mathbb{R}$, such that $x_k \to \infty$ as $k \to \infty$. Prove that:

$$\sum_{k=1}^{\infty} c_k \delta(\cdot - x_k) \tag{2}$$

converges in $\mathcal{D}'(\mathbb{R})$, for any sequence $\{c_k\} \subset \mathbb{R}$.

**Exercise 3. [10 points]** Let $\psi \in L^2(\mathbb{R}^d)$ such that $|x|\psi(x) \in L^2(\mathbb{R}^d)$ and $\psi \in H^1(\mathbb{R}^d)$. Let:

$$\begin{aligned} X &:= \langle \psi, x\psi \rangle \;, \qquad \Delta X^2 := \langle (x-X)\psi, (x-X)\psi \rangle \\ P &:= \langle \psi, -i\nabla\psi \rangle \;, \qquad \Delta P^2 := \langle (-i\nabla - P)\psi, (-i\nabla - P)\psi \rangle \;. \end{aligned} \tag{3}$$

(Notice that $x = (x_1, \ldots, x_d)$. We are using the notation: $\langle x\psi, x\psi \rangle = \sum_{j=1}^{d} \langle x_j\psi, x_j\psi \rangle$. Same for $-i\nabla$.) Prove that:

$$\Delta X^2 \Delta P^2 \geq \frac{d^2}{4} \qquad \text{[Uncertainty principle]} \tag{4}$$

*[Hint. Use that, by Cauchy-Schwarz inequality:*

$$\begin{aligned} \Delta X^2 \Delta P^2 &= \langle (x-X)\psi, (x-X)\psi \rangle \langle (-i\nabla - P)\psi, (-i\nabla - P)\psi \rangle \\ &\geq |\langle \psi, (-i\nabla - P)(x-X)\psi \rangle|^2 \;. \end{aligned} \tag{5}$$

*Then, using $AB = \frac{1}{2}\{A,B\} + \frac{1}{2}[A,B]$ with $\{A,B\} = AB + BA$ and $[A,B] = AB - BA$, argue that, for the purpose of a lower bound, in Eq. (5) it is enough to replace $(-i\nabla - P)(x-X)$ by $[(-i\nabla - P), (x-X)]$. Compute the commutator.]*

**Exercise 4. [10 points]** Let $\mathcal{H} = L^2([0,1])$. For $\psi \in \mathcal{H}$, consider:

$$T(\psi) := \int_0^{1/2} dx\, \psi(x) \;. \tag{6}$$

Prove that $T \in \mathcal{H}'$. Recall the definition of norm of $T$:

$$\|T\| := \sup_{\psi \in \mathcal{H}} \frac{|T(\psi)|}{\|\psi\|_{\mathcal{H}}} \;. \tag{7}$$

Compute $\|T\|$ using Riesz theorem.