Mathematical Quantum Theory
Exercise sheet 7
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Exercise 1. [10 points] Let $L^2(\mathbb{T})$ and $H^1(\mathbb{T})$ be the Hilbert spaces of periodic square integrable functions and functions with square integrable weak derivatives, respectively, with the inner products

$$\langle f, g\rangle_{L^2} = \int_{\mathbb{T}} \overline{f} g \, dx, \quad \langle f, g\rangle_{H^1} = \int_{\mathbb{T}} (\overline{f} g + \overline{f'} g') \, dx$$

Let $D : H^1(\mathbb{T}) \to L^2(\mathbb{T})$ be the derivative operator $D = d/dx$. Prove that

$$D^* = D(D^2 - 1)^{-1}$$

Exercise 2. [10 points]

1) Let $T : D(T) \to \mathcal{H}$ a liner operator. Show that T is closed if and only if $D(T)$ is complete with respect to the norm $\| \cdot \|_{\Gamma(T)}$ such that

$$\|x\|_{\Gamma(T)} := \|x\|_{\mathcal{H}} + \|Tx\|_{\mathcal{H}} \qquad \text{(graph norm)}$$

2) Show that every operator $T \in \mathcal{L}(\mathcal{H})$ is closable and that every finite-rank closable operator is bounded

Exercise 3. [10 points]

1) Consider $(-\Delta, H^2(\mathbb{R}))$, prove that $-\Delta$ is an unbounded operator on $L^2(\mathbb{R})$.
2) Consider $H := -\Delta + V(x)$ with $V \in C(\mathbb{R})$ and $V(x) \geq 0$, prove that $(H, C_c^\infty(\mathbb{R}))$ is a positive operator.

Exercise 4. [10 points] Let $\mathcal{H}$ be a Hilbert space with the norm $\| \cdot \|$ and let $| \cdot |$ be another norm on $\mathcal{H}$ such that

$$\|x\| \leq C|x| \quad \forall x \in \mathcal{H},\, C > 0.$$

Let $T : \mathcal{H} \to \mathcal{H}$ be a symmetric operator such that

$$|Tx| \leq k|x|, \quad \forall x \in H,$$

for some $k > 0$. Prove that

$$\|Tx\| \leq k\|x\|, \quad \forall x \in H$$