Mathematical Quantum Theory
Exercise sheet 8
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Exercise 1. [14 points] Let $\mathcal{H}$ be a Hilbert space, let $H \in \mathcal{L}(\mathcal{H})$ be self-adjoint, and assume that there is an orthonormal basis $(\varphi_n) \subset \mathcal{H}$ and $(\lambda_n) \subset \mathbb{R}$ such that $H\varphi_n = \lambda_n \varphi_n$ for all $n \in \mathbb{N}$. Let $f \in C(\sigma(H))$ be a continuous function defined on the spectrum $\sigma(H)$. For $\psi \in \mathcal{H}$, define:

$$f(H)\varphi := \sum_{n=1}^{\infty} f(\lambda_n) P_{\varphi_n} \psi$$

with P_{φ_n} the orthogonal projection on φ_n.

(a) Show that $\Phi : C(\sigma(H)) \to \mathcal{L}(\mathcal{H})$, $f \mapsto \Phi(f) = f(H)$ defines a functional calculus.

(b) Prove that Φ is continuous. *[Hint: being H bounded, $\sigma(H)$ is compact].*

Exercise 2. [14 points] Let $\mathcal{H}$ be a Hilbert space. Let $T \in \mathcal{L}(\mathcal{H})$, $T = T^*$. Prove the following statements.

(a) $T \leq \|T\| \mathbb{1}_{\mathcal{H}}$.

(b) If $T \geq 0$ then $\sigma(T) \subseteq [0, \|T\|]$. *[Hint: use that $T = T^*$ implies Ran$(T + z) = \mathcal{H}$, Ker$(T + z) = \{0\}$ for $z = \varepsilon, \|T\| + \varepsilon$, for all $\varepsilon > 0$.]*

(c) If $\sigma(T) \subset [r, R]$ for some $r, R \in \mathbb{R}$ with $R > r$, then $r\mathbb{1}_{\mathcal{H}} \leq T \leq R\mathbb{1}_{\mathcal{H}}$.

(d) If $T \leq S$ then $A^*TA \leq A^*SA$ for all $A \in \mathcal{L}(\mathcal{H})$.

(e) If $T \geq 0$, then T is invertible if and only if $T \geq c\mathbb{1}$ for some $c > 0$.

Exercise 3. [12 points] Let $(T, D(T))$ be a selfadjoint operator. Prove that:

$$\inf \sigma(T) = \inf_{\psi \in D(T): \|\psi\|=1} \langle \psi, T\psi \rangle \ , \qquad \sup \sigma(T) = \sup_{\psi \in D(T): \|\psi\|=1} \langle \psi, T\psi \rangle \ .$$

Exercise 4. [10 points, bonus] Let $(T, D(T))$ be a closed, densely defined, linear operator on $\mathcal{H}$. Prove that $\rho(T) \ni z \mapsto R_z(T) = (T - z)^{-1}$ is differentiable, and that:

$$\frac{d}{dz} R_z(T) = R_z(T)^2 \ . \tag{1}$$