Mathematical Quantum Theory
Exercise sheet 9
26.01.2019

Emanuela Giacomelli
emanuela-laura.giacomelli@uni-tuebingen.de

Exercise 1. [15 points] Consider the discrete Laplacian $-\Delta : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$, given by:

$$(-\Delta\psi)_n = \sum_{\substack{m\in\mathbb{Z}\\ |m-n|=1}} (\psi_n - \psi_m) = 2\psi_n - \psi_{n-1} - \psi_{n+1} \;, \qquad \forall n \in \mathbb{Z} \;.$$

(i) Show that $-\Delta$ is bounded and selfadjoint.

(ii) Show that $0 \le -\Delta \le 4\mathbb{1}$.

(iii) Compute $\| - \Delta\|$.

(iv) Determine $\sigma(-\Delta)$ using the Weyl criterion. *[Hint. Use that $-\Delta f = \lambda f$ for $f = e^{i\alpha n}$ and for some explicit $\lambda = \lambda(\alpha)$.]*

Exercise 2. [10 points] Let T be selfadjoint. Prove that, as $\varepsilon \to 0^+$:

$$\frac{1}{2\pi i}\int_a^b \Big((T-\lambda-i\varepsilon)^{-1} - (T-\lambda+i\varepsilon)^{-1}\Big) d\lambda \to \frac{1}{2}\big[\chi_{[a,b]}(T) + \chi_{(a,b)}(T)\big] \qquad \text{strongly,}$$

with $\chi_\Omega(T) \equiv \Phi(\chi_\Omega)$.

Exercise 3. [15 points] Let H be a bounded selfadjoint operator. Prove that

(i) The operator $U(t) = e^{iHt}$ defined via the functional calculus is unitary for all $t \in \mathbb{R}$ and:

$$U(t)^* = U(-t) \;, \qquad U(t)U(s) = U(t+s) \;, \qquad \forall t, s \in \mathbb{R} \;.$$

(ii) The operator-valued function $t \mapsto U(t)$ defined in (i) via the functional calculus is differentiable with respect to the operator norm topology, and $U'(t) = iHU(t)$ for all $t \in \mathbb{R}$.

(iii) For $\lambda \notin \sigma(H)$, we have:

$$\|(H-\lambda)^{-1}\| = \operatorname{dist}(\lambda, \sigma(H))^{-1} \;.$$