Introduction to Partial Differential Equations
Exercise sheet 1
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Emanuela Giacomelli
emanuela-laura.giacomelli@uni-tuebingen.de

Exercise 1. [10 points]

Write down an explicit formula for a function $u : \mathbb{R}^n \times [0, \infty)$ solving the initial value problem

$$\begin{cases} u_t + b \cdot Du + tu = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

where $b \in \mathbb{R}^n$ is a constant.

Exercise 2. [10 points]

Let $U \subset \mathbb{R}^n$ be an open set.

1. Given a subharmonic function $v \in C^2(\overline{U})$, i.e. such that

   $$-\Delta v \leq 0 \qquad \text{in } U,$$

   prove that

   $$v(x) \leq \frac{1}{|B(x,r)|} \int_{B(x,r)} dy\, v(y) \qquad \text{for all } B(x,r) \subset U.$$

   Prove also that if $v \in C^2(\overline{U})$ is subharmonic, then $\max_{\overline{U}} \tilde{v} = \max_{\partial U} \tilde{v}$.

2. Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume u is harmonic and $v := \phi(u)$. Prove that v is subharmonic.
3. Prove $v := |Du|^2$ is subharmonic, whenever u is harmonic.
4. Let u be harmonic in $\mathbb{R}^n$. Suppose that $\int_{\mathbb{R}^n} dx\, u(x)^2 \leq M < \infty$. Show that $u(x) \equiv 0$.

Remark 1 *It easily follows that if $v \in C^2(\overline{U})$ is superharmonic, i.e. $-\Delta v \geq 0$, then*

$$v(x) \geq \frac{1}{|B(x,r)|} \int_{B(x,r)} dy\, v(y) \qquad \textit{for all } B(x,r) \subset U,$$

and that $\min_{\overline{U}} \tilde{v} = \min_{\partial U} \tilde{v}$.

Exercise 3. [10 points]

Find the Green's function for the domain:

$$B_r^+ := \{x \in \mathbb{R}^2 \,|\, x_1^2 + x_2^2 < r, x_2 > 0\}.$$