Introduction to Partial Differential Equations
Exercise sheet 2
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Exercise 1. [10 points]

1. Let u be harmonic in $\mathbb{R}^n$. Suppose that
$$\int_{\mathbb{R}^n} dx\, |\partial_{x_i} u(x)|^p \leq M < +\infty,$$
with $p > 1$. Prove that u is constant in x_i. [*Hint: Recall Hölder inequality*].

2. Let $U \subset \mathbb{R}^n$ be an open, bounded and connected subset of $\mathbb{R}^n$, let u be harmonic in U. Suppose $u \geq 0$ in U and suppose that there exists a point $x_\star \in U$ such that $u(x_\star) > 0$. Prove that $u(x) > 0$ for all $x \in U$.

Exercise 2. [10 points]

Let $U \subset \mathbb{R}^n$ be a bounded, open subset of $\mathbb{R}^n$. Prove that there exists a constant C, depending on U such that
$$\max_{\overline{U}} |u| \leq C\big(\max_{\partial U} |g| + \max_{\overline{U}} |f|\big),$$
whenever u is a solution of
$$\begin{cases} -\Delta u = f & inU \\ u = g & onU, \end{cases}$$
[*Hint: use that the function* $u + \frac{|x|^2}{2n}\lambda$*, with* $\lambda := \max_{\overline{U}} |f|$*, is subharmonic*].

Exercise 3. [10 points]

1. Use Poisson's formula for the ball $B(0, r) \subset \mathbb{R}^n$ to prove
$$r^{n-2}\frac{r - |x|}{(r + |x|)^{n-1}} u(0) \leq u(x) \leq r^{n-2}\frac{r + |x|}{(r - |x|)^{n-1}} u(0),$$
whenever u is poitive and harmonic in $B^0(0, r) := \big\{x \in \mathbb{R}^n \text{ s.t. } |x| < r\big\}$. This is an explicit form of Harnack's inequality.

2. Let u be the solution of
$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+ \\ u = g & \text{on } \partial\mathbb{R}^n_+, \end{cases}$$
with $g \in C(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$, then u is given by the Poisson's formula. Assume $g(x) := |x|$ for $x \in \partial\mathbb{R}^n_+$, if $|x| \leq 1$. Show that Du is not bounded near $x = 0$. [*Hint: estimate* $\frac{u(\lambda e_n) - u(0)}{\lambda}$].