Introduction to Partial Differential Equations
Exercise sheet 4
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Exercise 1. [10 points]

Let $U \subset \mathbb{R}^n$ open and bounded, let $U_T := U \times (0, T]$ where $T > 0$ is fixed. We say that $v \in C_1^2(U_T) \cap C(\overline{U}_T)$ is a subsolution of the heat equation if

$$v_t - \Delta v \leq 0 \quad \text{in } U_T.$$

1) Prove for a subsolution v that

$$v(x,t) \leq \frac{1}{4r^n} \int\int_{E(x,t;r)} v(y,s) \frac{|x-y|^2}{(t-s)^2} dyds,$$

for all $E(x,t;r) \subset U_T$, where for every fixed $x \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $r > 0$, we define

$$E(x,t;r) := \left\{ (y,s) \in \mathbb{R}^{n+1} \,|\, s \leq t,\ \Phi(x-y, t-s) \geq \frac{1}{r^n} \right\}$$

2) Prove that

$$\max_{\overline{U}_T} u = \max_{\Gamma_T} u,$$

where $\Gamma_T := \overline{U}_T - U_T$

Exercise 2. [10 points]

1) Let $\Phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume that u solves the heat equation and let $v := \Phi(u)$. Prove that v is a subsolution.
2) Let $v := |D_x u|^2 + u_t^2$, with u a smooth solution of the heat equation. Prove that v is a subsolution.

Exercise 3. [10 points]

Given $g : [0, \infty) \to \mathbb{R}$, with $g(0) = 0$, derive the formula

$$u(x,t) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t-s)^{3/2}} e^{-\frac{x^2}{4(t-s)}} g(s)\, ds$$

for a solution of the initial/boundary-value problem

$$\begin{cases} u_t - u_{xx} = 0 & \text{in } \mathbb{R}_+ \times (0,\infty) \\ u(x,0) = 0 & \text{for } x \in \mathbb{R}_+ \\ u(0,t) = g(t) & \text{for } t \in [0,\infty) \end{cases}$$

[*Hint: define $v(x,t) := u(x,t) - g(t)$ and extend v to $\{x < 0\}$ by odd reflection.*]