Introduction to Partial Differential Equations
Exercise sheet 6, 17.12.2018
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Exercise 1. [10 points]

1) Let $U \subset \mathbb{R}^n$ bounded, with ∂U of class C^1. Consider the initial value/boundary problem:

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } U \times (0,\infty), \\ u(x,0) = g(x),\ u_t(x,0) = h(x) & \text{for } x \in U, \\ \nu \cdot Du + a(x) u_t = 0 & \text{on } \partial U. \end{cases}$$

with $\nu \equiv \nu(x)$ the outward normal at $x \in \partial U$. Suppose that $a(x) \geq 0$. Define the energy functional for the problem, and prove uniqueness of the solution.

2) Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Consider the initial- boundary value problem

$$\begin{cases} u_{tt} + a^2(x,t) u_t - \Delta u = 0 & \text{in } \Omega \times (0,\infty), \\ u(x,t) = 0, & \text{on } \partial\Omega,\ 0 < t < +\infty, \\ u(x,0) = f(x),\ u_t(x,0) = g(x) & \text{for } x \in \Omega. \end{cases}$$

where $a(x,t)$, $f(x)$, $g(x)$ are smooth functions. Let

$$E(t) := \int_\Omega (u_t^2 + |\nabla u|^2)\, \mathrm{d}x.$$

Prove that $E(t)$ is a decreasing function of t and prove that the L^2-norm of the solution is bounded, i.e., $\|u(\cdot,t)\|_{L^2(\Omega)} < \infty$.

Exercise 2. [10 points]

Let $U \subset \mathbb{R}^n$ be open and bounded. Consider the initial value/boundary problem

$$\begin{cases} \varepsilon^2 u_{tt} + u_t = \Delta u & \text{in } U \times (0,\infty), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,\infty). \end{cases} \tag{1}$$

1) Show that the energy functional

$$E(t) := \int_U \varepsilon^2 u_t^2 + |\nabla u|^2\, dx$$

is nonincreasing on solution of the problem (1).

2) Consider now the solution $u \equiv u^\varepsilon$ of (1) with intial datum $u^\varepsilon(x,0) = 0$, $u_t^\varepsilon(x,0) = \varepsilon^{-\alpha} f(x)$, with f independent on f and $\alpha < 1$. Prove that

$$\int_U |\nabla u^\varepsilon(x,t)|^2\, \mathrm{d}x \to 0 \quad \text{as } \varepsilon \to 0.$$

Exercise 3. [10 points]
Consider the nonhomogeneous Burgers' equation:

$$\begin{cases} \partial_t u + u \partial_x u = -x & \text{for } (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = f(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

Find the solution using the method of characteristic (that is, find an implicit expression for u, such that for small times gives the unique solution).