Introduction to Partial Differential Equations
Exercise sheet 3
26.11.2018

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Exercise 1. [10 points]

1. Let $B_1^+ \subset \mathbb{R}^2$ denote the upper half-disc, $B_1^+ = \{(x,y) \in \mathbb{R}^2 \,|\, x^2+y^2 < 1, y > 0\}$. Take $u \in C^2(B_1^+) \cap c(\overline{B_1^+})$ harmonic on B_1^+ with $u(x,0) = 0$. Prove that the continuous function
$$U(x) = \begin{cases} u(x,y) & \text{if } y \geq 0 \\ -u(x,-y) & \text{if } y < 0 \end{cases}$$
obtained through an odd reflection of u with respect to the $x-$th axis, is harmonic on the entire disc $B_1 = \{(x,y) \in \mathbb{R}^2 \,|\, x^2+y^2 < 1\}$.

2. Solve the following problem
$$\begin{cases} \Delta u(x,y) = 0 & \text{in } B_1^+ \\ u = g & \text{on } \partial B_1^+, \end{cases}$$
with g continuous on ∂B_1.

Exercise 1. [10 points]
Write down an explicit formula for a solution of
$$\begin{cases} u_t - \Delta u + cu = & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\}, \end{cases}$$
where $c \in \mathbb{R}$.

Exercise 3. [10 points]

Write down an explicit formula for a solution $u(x,t)$ of
$$\begin{cases} u_t - u_{xx} = 0 & \text{if } 0 < x < L,\ t > 0 \\ u(x,0) = g(x) & \text{if } 0 \leq x \leq L \\ u_x(0,t) = u_x(L,t) = 0 & \text{if } t > 0, \end{cases}$$
where g is a continuous function in $[0,L]$.

[*Hint: find the solution by reflection, as in the case of the half-line- To do so, define the even reflection of* g:
$$\tilde{g}(x) := \begin{cases} g(x) & \text{if } 0 \leq x \leq L \\ g(-x) & \text{if } -L \leq x < 0 \end{cases}$$
Then, extend $\tilde{g}$ *to the whole* $\mathbb{R}$ *by setting it to zero outside the interval* $[-L,L]$. *Finally, define:*
$$g^\star(x) = \sum_{n=-\infty}^{+\infty} \tilde{g}(x-2nL)$$
Notice that for every fixed $x \in \mathbb{R}$, *there is only one non vanishing term in the sum.*]