Universität Tübingen

Fachbereich Mathematik

Introduction to Partial Differential Equations

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Introduction

These notes are the transcription of the lectures of the course *Introduction to Partial Differential Equations* given by Marcello Porta during the academic year 2017-2018, at the University of Tübingen. They are *not* meant to replace an ordinary textbook, but rather to help the students in keeping track of the topics discussed in class. The list of recommended textbooks can be found on the course webpage: https://www.math.uni-tuebingen.de/arbeitsbereiche/maphy/lehre/ws-2017-18/DiffEquat.

The reader is encouraged to point out mistakes and typos, that will unavoidably be present, and to report them to: giovanni.antinucci@math.uzh.ch.

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Chapter 1

Four important PDEs

1.1 Introduction, notation and definitions

Definition 1 (Derivatives)**.** *Let $U \subseteq \mathbb{R}^n$ an open subset of $\mathbb{R}^n$, $x \in U$ and $u : U \to \mathbb{R}$. Let $\alpha = \{\alpha_1, \cdots, \alpha_n\}$ be a multi-index ($\alpha_i \in \mathbb{N}$). Then:*

$$D^\alpha u(x) := \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n} u(x), \qquad |\alpha| = \sum_{i=1}^{n} \alpha_i. \tag{1.1}$$

In the following, we will also use the notation

$$\partial_{x_i} \equiv \frac{\partial}{\partial x_i}, \qquad u_{x_i} \equiv \partial_{x_i} u. \tag{1.2}$$

Definition 2 (Set of partial derivatives of order K)**.** *Let $K \in \mathbb{N}$. Then:*

$$D^K u := \left\{ D^\alpha u \;\middle|\; |\alpha| = K \right\}. \tag{1.3}$$

We shall denote by $C^K(U)$ the set:

$$C^K(U) := \left\{ u : U \to \mathbb{R} \;\middle|\; u \text{ is } K-\text{times continuously differentiable} \right\} \tag{1.4}$$

Example 1.1.1.

- $Du = (\partial_{x_1} u, \cdots, \partial_{x_n} u) \equiv \nabla u$ is the gradient.
- Let $\alpha = (\alpha_1, 0, 0, \cdots, 0)$. Then: $D^\alpha \equiv \partial_{x_1}^{\alpha_1} u$.
- Let $\alpha = (\alpha_1, \alpha_2, 0, \cdots, 0)$. Then $D^\alpha u = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} u$.

Remark 1.1. *$D^K u(x)$ can be thought as a point in $\mathbb{R}^{n^K}$ for any fixed $x \in U$. Indeed, the number of element of $D^K u$ is*

$$\sum_{\substack{\alpha_1, \cdots, \alpha_n \\ \sum_{i=1}^n \alpha_i = K}} \binom{K}{\alpha_1 \cdots \alpha_n} \underbrace{=}_{\textit{multinomial theorem}} n^K, \tag{1.5}$$

where

$$\begin{aligned}\binom{K}{\alpha_1 \cdots \alpha_n} = &\ \textit{number of ways of} \\ &\ \textit{partitioning } K \textit{ objects in } \{\alpha_1, \alpha_2, \cdots, \alpha_n\} \textit{ s.t.} \sum_i \alpha_i = K.\end{aligned} \tag{1.6}$$

Definition 3 (Partial Differential Equation)**.** *Let*

$$F : \mathbb{R}^{n^K} \times \mathbb{R}^{n^{K-1}} \times \cdots \times \mathbb{R}^n \times R \times U \longrightarrow \mathbb{R}. \tag{1.7}$$

A partial differential equation (PDE) is an equation of the form

$$F\left(D^k u(x), D^{k-1}u(x), \cdots, Du(x), u(x), x\right) = 0, \qquad x \in U. \tag{1.8}$$

Definition 4 (Classification of PDEs)**.** *We say that the PDE is*

- **linear** *if it has the form*
$$\sum_{|\alpha|=K} a_\alpha(x) D^\alpha u(x) = f(x), \tag{1.9}$$
for given functions a_α and f.

- **semi-linear** *if*
$$\sum_{|\alpha|_K} a_\alpha(x) D^\alpha u + a_0(D^{K-1}u, \cdots, Du, u, x) = 0, \tag{1.10}$$
i.e. it is linear in the higher derivatives only.

- **quasi-linear** *if*
$$\sum_{|\alpha|=K} a_\alpha(D^{K-1}u, \cdots, u, x) D^\alpha u + a_0(D^{k-1}u, \cdots, Du, u, x = 0). \tag{1.11}$$
A semi-linear PDE is in fact a particular quasi-linear PDE.

- **fully non-linear** *if it depends non-linearly upon the higher derivatives.*

More generally, one could introduce systems of PDEs, by considering:

$$\begin{aligned} F &\longrightarrow \underline{F} = (F_1, \cdots, F_m), \\ u &\longleftrightarrow \underline{u} = (u_1, \cdots, u_m). \end{aligned} \tag{1.12}$$

Definition 5 (Order of a PDE)**.** *The order of a PDE is the order of the highest derivative of u appearing in the PDE.*

We shall discuss some methods to study the solutions of PDEs. In general, systems are more complicated to study with respect to equations, while non-linear PDEs are more difficult than the linear ones. It is worth also remarking that *there is no general theory* to solve PDEs; instead, there exist methods for *classes of PDEs*.
A typical question one is interested in is to establish existence of solutions for a given PDE. Then, one might investigate regularity, uniqueness, etc. Later, we shall see that the notion of "solution" of a PDE will have to be generalized, in order to take into account weak solutions as well.

1.2 Transport equation

1.2.1 Homogeneous transport equation

Given $u \equiv u(x,t), x \in \mathbb{R}^n, t \in (0,\infty)$, the transport equation is a linear, first order PDE:

$$\partial_t u + b \cdot Du = 0. \tag{1.13}$$

Let us consider the initial value problem:

$$\begin{cases} u_t + b \cdot Du = 0 & (x,t) \in \mathbb{R}^n \times (0,\infty), \\ u = g & (x,t) \in \mathbb{R}^n \times \{0\}, \end{cases} \tag{1.14}$$

where $g \equiv g(x)$ is a given function and $b \in \mathbb{R}^n$ is a fixed vector.

Eq. (1.14) is telling us that a certain directional derivative is vanishing. Let us define:

$$z(s) = u(x + sb, t + s), \qquad (x,t) \text{ fixed.} \tag{1.15}$$

One can easily check that:

$$u_t + b \cdot Du = 0 \Leftrightarrow \frac{d}{ds} z(s) = 0, \tag{1.16}$$

meaning that $u(s + bt, t + s)$ is constant in s. Thus, choosing $s_0 = -t$, and recalling the boundary condition of (1.14), we get:

$$z(s_0) = u(x,t) = u(x - t_b, 0) = g(x - tb). \tag{1.17}$$

The solution corresponds to the rigid transport of the graph of the initial datum, along the constant rectilinear motion.

1.2.2 Non-homogeneous transport equation

Consider now the nonhomogeneous initial value problem:

$$\begin{cases} u_t + b \cdot Du = f & (x,t) \in \mathbb{R}^n \times (0,\infty), \\ u = g & (x,t) \in \mathbb{R}^n \times \{0\}, \end{cases} \tag{1.18}$$

where $f \equiv f(x,t)$ and $g \equiv g(x)$ given, and $b \in \mathbb{R}^n$.

As before, define $z(s)$ as in (1.15). Using the first of (1.18), we get

$$\frac{d}{ds} z(s) = f(x + sb, t + s). \tag{1.19}$$

Therefore,

$$\begin{aligned} u(x,t) - u(x - tb, 0) =& z(0) - z(-t) = \int_{-t}^{0} ds \frac{d}{ds} z(s) = \\ &= \int_{-t}^{0} ds f(x + sb, t + s) \equiv \int_0^t ds f(x + (s - tb), 0), \end{aligned} \tag{1.20}$$

which implies that

$$u(x,t) = g(x - tb) + \int_0^t ds f(x + (s - t)b, 0). \tag{1.21}$$

1.3 Laplace and Poisson equations

1.3.1 Laplace equation

Let the Laplacian Δ be defined as

$$\Delta = \sum_{i=1}^{n} \partial_{x_i}^2. \tag{1.22}$$

Definition 6 (Laplace equation and harmonic functions)**.** *The Laplace equation is:*

$$\Delta u = 0. \tag{1.23}$$

A solution of the Laplace equation is called harmonic function.

Proposition 1. *The Laplace equation is invariant under rotations. That is, let $u(x)$ be a solution of $\Delta u = 0$. Then $v(x) = u(Rx)$ with $RR^T = R^T R = 1$ solves $\Delta v = 0$.*

Proof. Let $y := Ox$, where O is the $n \times n$ orthogonal matrix whose entries are $\{a_{ij}\}_{1 \leqslant i,j \leqslant n}$. So

$$v(x) = u(Ox) = u(y), \tag{1.24}$$

where $y_j = \sum_{i=1}^{n} a_{ji} x_i$. We compute

$$\frac{\partial v}{\partial x_i} = \sum_{j=1}^{n} \frac{\partial u}{\partial y_j} \frac{\partial y_j}{\partial x_i} = \sum_{j=1}^{n} \frac{\partial u}{\partial y_j} a_{ji}, \tag{1.25}$$

which implies

$$\begin{pmatrix} \frac{\partial v}{\partial x_1} \\ \vdots \\ \frac{\partial v}{\partial x_n} \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{n1} \\ \vdots & & \vdots \\ a_{1n} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial y_1} \\ \vdots \\ \frac{\partial u}{\partial y_n} \end{pmatrix} = O^T \begin{pmatrix} \frac{\partial u}{\partial y_1} \\ \vdots \\ \frac{\partial u}{\partial y_n} \end{pmatrix}, \tag{1.26}$$

that is:

$$D_x \cdot v = O^T D_y \cdot u. \tag{1.27}$$

Finally

$$\begin{aligned} \Delta v = D_x v \cdot D_x v = (O^T D_y u) \cdot (O^T D_y u) = \\ = (O^T D_y u)^T O^T D_y u = (D_y u)^T (O^T)^T O^T D_y u = \\ = (D_y u)^T O O^T D_y u = (D_y u)^T D_y u = (D_y u) \cdot (D_y u) = \\ = \Delta u(y) = 0. \end{aligned} \tag{1.28}$$

□

Remark 1.2. *The Laplace equation is linear. Suppose u_1, u_2 are two harmonic functions in $U \subset \mathbb{R}^n$. Then, $\Delta(u_1 + u_2) = 0$.*

Motivations. The Laplace equation describes the equilibrium state of incompressible fluids. Let $U \subset \mathbb{R}^n$, and let $u : \bar{U} \to \mathbb{R}$ be the the density of a given fluid. Let us now consider the subset $V \subset U$, and introduce the vector field F, describing how much fluid passes through the boundary ∂V of the given region V. One expects the fluid to move from regions of higher concentration towards regions of lower concentration. It is reasonable to assume that:

$$F(x) = -\nabla u(x). \tag{1.29}$$

If the fluid is incompressible, the amount of fluid entering V is equal to the amount of fluid exiting V. Thus, by Gauss theorem,

$$0 = \int_{\partial V} F \cdot \nu \, dS = \int_V \mathrm{div} F dx \tag{1.30}$$

where $\nu \equiv \nu(x)$ is the outward normal and $\mathrm{div} F = \partial_{x_1} F_1 + \cdots + \partial_{x_n} F_n$ is the divergence of F. Being V is arbitrary, we get:

$$\mathrm{div} F \equiv \mathrm{div} \nabla u = 0, \tag{1.31}$$

or, equivalenly, $\Delta u = 0$.

1.3.2 Poisson equation

Definition 7 (Poisson equation)**.** *Let $U \subset \mathbb{R}^n$ be an open domain, and $u : \bar{U} \to \mathbb{R}^n$. The Poisson equation is*

$$-\Delta u = f. \tag{1.32}$$

What does the Poisson equation describe? In classical electrodynamics, the *electric field* $E(x)$ solves the Maxwell equation:

$$\nabla \cdot E = 4\pi\rho, \tag{1.33}$$

where $\rho(x)$ is the density of charge. Let $u(x)$ be the electrostatic potential, in terms of which $E = -\nabla u$. Then,

$$-\Delta u = 4\pi\rho. \tag{1.34}$$

Setting $4\pi\rho(x) \equiv f(x)$, we get (1.32).

1.3.3 Fundamental solution

Laplace's equation. Being the Laplace equation invariant under rotations, recall Proposition 1, it is natural to look for solutions that enjoy this symmetry. That is, we are looking for solutions that only depend on $r \equiv |x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$,

$$u(x) = v(r). \tag{1.35}$$

With the purpose of writing explicitly $\Delta u = 0$ in radial coordinates, let us compute

$$\begin{aligned} u_{x_i} &= \partial_{x_1} u(x) = \partial_{x_i} v(|x|) = v'(|x|) \frac{x_i}{|x|}, \\ u_{x_i x_i} &= v''(|x|) \left(\frac{x_i}{|x|} \right)^2 + v'(|x|) \left(\frac{1}{|x|} - \frac{x_i^2}{|x|^3} \right). \end{aligned} \tag{1.36}$$

Therefore,

$$\sum_{i=1}^{n} u_{x_i x_i} = v''(r)\frac{r^2}{r^2} + v'(r)\left(\frac{n}{r} - \frac{r^2}{r^3}\right) \tag{1.37}$$

meaning that

$$\Delta u = 0 \Rightarrow v''(r) + v'(r)\frac{n-1}{r} = 0. \tag{1.38}$$

By defining $h(r) = v'(r)$, Eq. (1.38) becomes:

$$h'(r) + h(r)\frac{n-1}{r} = 0 \Rightarrow h(r) = \frac{a}{r^{n-1}}, \tag{1.39}$$

which gives:

$$v(r) = \int dr h(r) = \int dr \frac{a}{r^{n-1}} = \begin{cases} b(\log r) + c & \text{if } n = 2, \\ \frac{b}{r^{n-2}} + c & \text{if } n \geqslant 3. \end{cases} \tag{1.40}$$

Definition 8 (Fundamental solution of the Laplace equation)**.** *For $x \in \mathbb{R}^n \backslash \{0\}$, the function*

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \log |x| & \textit{if } n = 2, \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}} & \textit{if } n \geqslant 3, \end{cases} \tag{1.41}$$

with $\alpha(n) = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}$ the volume of the unit ball in $\mathbb{R}^n$, $\Gamma(z) = \int_0^\infty dx x^{z-1} e^{-x}$, is called the fundamental solution of the Laplace equation*.*

Poisson's equation. We will use the *fundamental solution* to find the solution of Poisson's equation $-\Delta u = f$. Consider the function:

$$\int dy\, \Phi(x-y) f(y). \tag{1.42}$$

Being $\Phi(x)$ harmonic for $x \neq 0$, $\Phi(x-y)$ is harmonic for $x \neq y$; thus, $\Phi(x-y)f(y)$ is still harmonic for $x \neq y$. Nevertheless, the convolution (1.42), which can be read as a linear combination of solution of the Laplace equation, *is not harmonic*. To understand why, we formally compute:

$$\Delta u = \int dy \Delta\Phi(x-y) f(y), \tag{1.43}$$

which is however problematic, since $\Delta\Phi(x-y) \simeq \frac{1}{|x-y|^n}$, which is not integrable at $x = y$. Instead, we shall see that the function defined in Eq. (1.42) is a solution of the Poisson equation.

Theorem 1.1 (Solution of the Poisson equation)**.** *Let $f \in C_c^2(\mathbb{R}^n)$, and let*

$$u(x) = \int dy\, \Phi(x-y) f(y). \tag{1.44}$$

Then, $u \in C^2(\mathbb{R}^n)$, and $-\Delta u = f$ in $\mathbb{R}^n$.

Remark 1.3. *This theorem provides one solution of the Poisson problem in $\mathbb{R}^n$. We still do not know anything about uniqueness.*

Proof. The proof consists in four steps: the explicit computation of $u_{x_i x_j}$, the rewriting of Δu as the sum of two integrals: the dominating part and the remainder, and finally the explicit estimates for these two integrals.

1. Let
$$u(x) = \int_{\mathbb{R}^n} dy\, \Phi(x-y) f(y) = \int_{\mathbb{R}^n} dy\, \Phi(y) f(x-y), \tag{1.45}$$
by a simple change of variables. Let $e_1, \cdots, e_n$ be the standard basis of $\mathbb{R}^n$, and consider
$$\frac{u(x+he_i) - u(x)}{h} = \int_{\mathbb{R}^n} dy\, \Phi(y) \frac{f(x+he_i - y) - f(x-y)}{h}. \tag{1.46}$$
By assumption $f \in C_c^2$, which means that
$$\lim_{h\to 0} \frac{f(x+he_i-y) - f(x-y)}{h} = f_{x_i}(x-y), \tag{1.47}$$
and the limit is reached uniformly in $x-y$. Therefore,
$$u_{x_i}(x) = \int_{\mathbb{R}^n} \Phi(y) f_{x_i}(x-y) dy. \tag{1.48}$$
Repeating the argument,
$$u_{x_i x_j} = \int_{\mathbb{R}^n} \Phi(y) f_{x_i x_j}(x-y) dy, \tag{1.49}$$
and using again that $f \in C_c^2$ we infer that $u_{x_i x_j}$ is continuous in x.

2. Let us now compute the integral. Let $\epsilon > 0$, and let us rewrite:
$$\Delta u(x) = \underbrace{\int_{B(0,\epsilon)} \Phi(y) \Delta_x f(x-y) dy}_{I_\epsilon} + \underbrace{\int_{\mathbb{R}^n \setminus B(0,\epsilon)} \Phi(y) \Delta_x f(x-y) dy}_{J_\epsilon}. \tag{1.50}$$
We will study the two integrals separately.

3. We have:
$$\begin{aligned} |I_\epsilon| &\leqslant \int_{B(0,\epsilon)} |\Phi(y)|\, |\Delta_x f(x-y)|\, dy \leqslant \\ &\leqslant \sup_{y\in B(0,\epsilon)} |\Delta_x f(x-y)| \int_{B(0,\epsilon)} |\Phi(y)|\, dy \end{aligned} \tag{1.51}$$
Being $D^2 f$ continuous and compactly supported in $B(0,\epsilon)$, it is bounded by a constant. Hence:
$$(1.51) \leqslant C \int_{B(0,\epsilon)} |\Phi(y)|\, dy \leqslant C \begin{cases} \int_{B(0,\epsilon)} dy |\log|y||, & \text{if } n = 2, \\ \int_{B(0,\epsilon)} dy \frac{1}{|y|^{n-2}} & \text{if } n \geqslant 3. \end{cases} \tag{1.52}$$

Now we are left with bounding the integrals: let us first study the case $n = 2$.

$$\begin{aligned} \int_{B(0,\epsilon)} d^2y\,|\log|y|| = &\left|\int_{B(0,\epsilon)} d^2y \log|y|\right| = c\left|\int_0^\epsilon dr r\log r\right| = \\ =c&\left|\frac{r^2}{2}\log r\Big|_0^\epsilon - \int_0^\epsilon dr \frac{r^2}{2}\frac{1}{r}\right| \leqslant c\left(\epsilon^2\,|\log\epsilon| + \epsilon^2\right) \xrightarrow[\epsilon\to 0]{} 0 \end{aligned} \tag{1.53}$$

Suppose now $n \geqslant 3$.

$$\int_{B(0,\epsilon)} dy \frac{1}{|y|^{n-2}} = c\int_0^\epsilon dr \frac{r^{n-1}}{r^{n-1}} = c\int_0^\epsilon dyr = c\epsilon^2 \xrightarrow[\epsilon\to 0]{} 0 \tag{1.54}$$

We then conclude that $I_\epsilon \xrightarrow[\epsilon\to 0]{} 0$.

4. In order to study the integral J_ϵ, it is worth recalling that, by Gauss-Green:

$$\int_U u_{x_i} dx = \int_{\partial U} u\nu_i ds \tag{1.55}$$

where ν is the outward normal of ∂U. Let $u = \Phi(y)\partial_{y_i} f(x-y)$. We get:

$$\int_{B(0,\epsilon)} dy\left[\Phi(y)\partial^2_{y_i} f(x-y) + \partial_{y_i}\Phi(y)\partial_{y_i} f(x-y)\right] = \int_{\partial B(0,\epsilon)} \nu_i\Phi(y)\partial_{y_i} f(x-y)ds(y) \tag{1.56}$$

so that

$$J_\epsilon = \underbrace{\int_{\partial B(0,\epsilon)} \Phi(y)\nu\cdot D_y f(x-y)ds(y)}_{L_\epsilon} - \underbrace{\int_{\mathbb{R}^n\setminus B(0,\epsilon)} dyD\Phi(y)\cdot D_y f(x-y)}_{K_\epsilon}. \tag{1.57}$$

We will study the two terms separately:

$$|L_\epsilon| \leqslant c\int_{\partial B(0,\epsilon)} |\Phi(y)| = ds(y)\begin{cases} c\epsilon|\log\epsilon| & \text{if } n=2, \\ c\epsilon & \text{if } n\geqslant 3. \end{cases} \tag{1.58}$$

Therefore, $L_\epsilon \xrightarrow[\epsilon\to 0]{} 0$. Integrating again by parts the last term K_ϵ we get

$$\begin{aligned} K_\epsilon &= \underbrace{\int_{\mathbb{R}^n\setminus B(0,\epsilon)} \Delta\Phi(y) f(x-y)dy}_{=0} - \int_{\partial B(0,\epsilon)} \nu\cdot D_y\Phi(y)f(x-y)ds(y) = \\ &= -\int_{\partial B(0,\epsilon)} \nu\cdot D_y\Phi(y)f(x-y)ds(y) \end{aligned} \tag{1.59}$$

where we used that $\Phi(y)$ is harmonic if $y \neq 0$. To evaluate the last term, we start by computing

$$\begin{aligned} D\Phi(y) &= D\begin{cases} -\frac{1}{2\pi}\log|y| & \text{if } n=2 \\ \frac{1}{n(n-2)\alpha(n)}\frac{1}{|x|^{n-2}} & \text{if } n\geqslant 3 \end{cases} = \\ &= \begin{cases} -\frac{1}{2\pi}\frac{y}{|y|^2} & \text{if } n=2, \\ -\frac{1}{n(n-2)\alpha(n)}(n-2)\frac{1}{|y|^{n-1}}\frac{y}{|y|} = -\frac{1}{n\alpha(n)}\frac{y}{|y|^n} & \text{if } n\geqslant 3, \end{cases} \end{aligned} \tag{1.60}$$

where we recall

$$\alpha(n) = \frac{\pi^{n/2}}{\Gamma(n/2+1)} \Rightarrow \alpha(2) = \pi .$$

Summarizing

$$D\Phi(y) = -\frac{1}{n\alpha(n)} \frac{y}{|y|^n}, \qquad \text{if } y \neq 0. \tag{1.61}$$

Moreover, the normal of $\partial B(0,\epsilon)$ is

$$\nu = -\frac{y}{|y|} = -\frac{y}{\epsilon} \Rightarrow \nu \cdot D\Phi(y) = \frac{y}{\epsilon} + \frac{1}{n\alpha(n)} \frac{y}{\epsilon^n} = \frac{1}{n\alpha(n)} \frac{1}{\epsilon^{n-1}}. \tag{1.62}$$

Hence

$$\begin{aligned} K_\epsilon &= -\int_{\partial B(0,\epsilon)} \nu \cdot D_y\Phi(y) f(x-y) ds(y) = \\ &= -\frac{1}{\epsilon^{n-1} n\alpha(n)} \int_{\partial B(0,\epsilon)} f(x-y) ds(y) \equiv \fint \partial B(x,\epsilon) f(y) ds(y) \xrightarrow[\epsilon \to 0]{} -f(x). \end{aligned} \tag{1.63}$$

Finally, taking the limit $\epsilon \to 0$ we get

$$\Delta u(x) = -f(x). \tag{1.64}$$

$\square$

Remark 1.4. *Even if the source term f is compactly supported, the fundamental solution Φ propagates it on all $\mathbb{R}^n$.*

Poisson's boundary problem. Let us now consider the *boundary value problem:*

$$\begin{cases} -\Delta u = f & \text{on } U, \\ u = g & \text{in } \partial U. \end{cases} \tag{1.65}$$

1.3.4 Mean-value formula

Theorem 1.2 (Mean value formula)**.** *Let $u \in C^2(U)$ be harmonic. Then*

$$u(x) = \fint_{\partial B(x,r)} u ds = \fint_{B(x,r)} u dy \,, \qquad \forall B \subset U \,. \tag{1.66}$$

Proof. Define

$$\phi(r) := \fint_{\partial B(x,r)} u(y) ds(y) = \fint_{\partial B(0,1)} u(x+rz) ds(z). \tag{1.67}$$

We compute:

$$\begin{aligned} \phi'(r) =& \fint_{\partial B(0,1)} Du(x+rz) \cdot z ds(z) = \fint_{\partial B(0,1)} Du(y) \frac{y-z}{r} ds(y) = \\ =& \frac{r}{n} \fint_{B(x,r)} \Delta u(y) dy = 0, \end{aligned} \tag{1.68}$$

being u harmonic. This means that ϕ is constant, which implies:

$$\phi(r) = \lim_{t\to 0} \phi(t) = \lim_{t\to 0} ⨍_{\partial B(x,t)} u(y) ds(y) = u(x). \tag{1.69}$$

To prove the equivalence with the average over the ball we write:

$$\begin{aligned}\int_{B(x,r)} u dy =& \int_0^r \left(\int_{\partial B(x,s)} u dS \right) ds = u(x) \int_0^r n\alpha(n) s^{n-1} ds = u(x)\alpha(n) r^n = \\ =& u(x)|B(x,r)|,\end{aligned} \tag{1.70}$$

which implies:

$$⨍_{B(x,y)} u dy = u(x) \ . \tag{1.71}$$

□

One can also prove the following converse implication.

Theorem 1.3 (Converse to mean value formula)**.** *Let $U \subset \mathbb{R}^n$, $u \in C^2(U)$, and suppose that*

$$u(x) = ⨍_{\partial B(x,r)} u ds, \qquad \forall B(x,r) \subset U. \tag{1.72}$$

Then u is harmonic.

Proof.

$$u(x) = ⨍_{\partial B(x,r)} u ds \Rightarrow \phi'(r) = 0. \tag{1.73}$$

Let us proceed by contradiction, supposing that u is not harmonic: $\Delta u(x) \neq 0$ for some $x \in U$. Then

$$0 = \phi'(r) = \frac{r}{n} ⨍_{B(x,r)} \Delta u(y) dy \neq 0 \tag{1.74}$$

which is a contradiction. □

Remark 1.5. *The mean-value formula is a very important property of harmonic functions. It is related to the Cauchy formula for analytic functions:*

$$f(x) = \frac{1}{2\pi i} \int_{\mathcal{C}(x)} dz \frac{f(z)}{z - x}, \tag{1.75}$$

with $\mathcal{C}(x)$ a closed curve, counterclockwise oriented, encircling the point x.

Application: Newton's theorem. Consider the Poisson problem, with f radial and compactly supported:

$$\begin{cases} -\Delta u = f & \text{on } \mathbb{R}^n, \\ f(x) \equiv f(|x|), & \\ f(x) = 0 & \text{if } |x| > R, \int dx f(x) < \infty. \end{cases} \tag{1.76}$$

Physically, the function f might describe a spherically symmetric charge distribution. In $\mathbb{R}^3$, the corresponding *electrostatic potential* is:

$$u(x) = \int_{\mathbb{R}^3} dy \frac{1}{|x-y|} f(y), \tag{1.77}$$

which is a solution of the Poisson problem.

Proposition 2. *Let $x \in \mathbb{R}^n$ s.t. $|x| > R$. Then*

$$u(x) = \int_{\mathbb{R}^n} dy \frac{1}{|x-y|} f(y) = \frac{1}{|x|} \underbrace{\int dy f(y)}_{\text{total charge of the ball}} . \tag{1.78}$$

Remark 1.6. *Thus, the potential generated by the spherical distribution f is equivalent, for $|x| > R$, to the potential generated by a point-like charge with charge $\int dy f(y)$.*

Proof. Let us rewrite, using the assumption that f is a radial function,

$$\begin{aligned} u(x) = \int_{|y| \leqslant R} dy f(y) \frac{1}{|x-y|} &= \int_0^R ds \left(\int_{\partial B(0,s)} dS(y) \frac{1}{|x-y|} \right) f(s) = \\ &= \int_0^R ds |\partial B(0,s)| \fint_{\partial B(0,s)} dS(y) \frac{1}{|x-y|} f(s). \end{aligned} \tag{1.79}$$

Noticing that $\frac{1}{|x-y|}$ is harmonic away from 0, we can use the mean value formula to rewrite

$$\fint_{\partial B(0,s)} dS(y) \frac{1}{|x-y|} = \frac{1}{|x|},$$

so that

$$(1.79) = \frac{1}{|x|} \int_0^R ds |\partial B(0,s)| f(s) \equiv \frac{1}{|x|} \int dy f(y). \tag{1.80}$$

$\square$

1.3.5 Maximum principle

Definition 9 (Subharmonic functions)**.** *A function u such that*

$$-\Delta u \leqslant 0, \tag{1.81}$$

is called subharmonic.

Theorem 1.4 (Weak maximum principle)**.** *Let $u \in C^2(U) \cap C(\bar{U})$, with $U \subset \mathbb{R}^n$ open and bounded. Suppose that u is subharmonic. Then*

$$\max_{u \in \bar{U}} u(x) = \max_{x \in \partial U} u(x). \tag{1.82}$$

Remark 1.7. *This means that a (sub-)harmonic function in an open bounded domain reaches its maximum value on the boundary.*

Proof. The proof is split in two parts: first we will assume that $-\Delta u < 0$, and then we will extend the proof to the harmonic functions.

1. Suppose that $-\Delta u < 0$ in U, and that there exists $x_0 \in U$ such that

$$u(x_0) = \max_{x \in \bar{U}} u(x). \tag{1.83}$$

Then, by definition of maximum

$$Du(x_0) = 0, \qquad D^2 u(x_0) \leqslant 0, \tag{1.84}$$

i.e. the gradient vanishes and the Hermitian matrix $H(x_0) = \{\partial_{x_i x_j} u(x_0)\}_{1 \leqslant i \leqslant j \leqslant n} \equiv D^2 u(x_0)$ is non-positive, meaning that

$$(\alpha, D^2 u(x_0)\alpha) \leqslant 0, \qquad \forall \alpha \in \mathbb{R}^n. \tag{1.85}$$

Equivalently, one can say that all the eigenvalues of $D^2 u(x_0)$ are non-positive. In particular this implies that

$$\Delta u(x_0) = Tr D^2 u(x_0) \leqslant 0, \tag{1.86}$$

which is a contradiction, since we assumed $-\Delta u < 0$. This means that the maximum of u is on ∂U.

2. Suppose now that

$$\Delta u = 0 \text{ on } U. \tag{1.87}$$

Let $v(x) = x^2$, then

$$-\Delta(u + \epsilon v) = -\Delta u - 2\epsilon < 0 \qquad \forall \epsilon > 0. \tag{1.88}$$

Therefore, we are back to point 1. We have

$$\max_{x \in U}(u + \epsilon v) = \max_{x \in \partial U}(u + \epsilon v). \tag{1.89}$$

Using that

$$\begin{aligned} \max_{x \in U} u + \epsilon \min_{x \in U} v \leqslant \max_{x \in U} |u + \epsilon v| \Rightarrow \max_{x \in \partial U}(u + \epsilon v) \leqslant \max_{x \in \partial U} u + \epsilon \max_{x \in \partial U} v \Rightarrow \\ \Rightarrow \max_{x \in \bar{U}} u + \epsilon_{x \in \bar{U}} v \leqslant \max_{x \in \partial U} u + \epsilon_{x \in \partial U} v. \end{aligned} \tag{1.90}$$

Since U is a bounded set, there exist two constants C, c such that

$$|\min_{x \in \bar{U}} v| \leqslant C, \qquad |\max_{x \in \partial U} v| \leqslant c, \tag{1.91}$$

so that, taking the limit $\epsilon \to 0$ we get

$$\max_{x \in \bar{U}} u \leqslant \max_{x \in \partial U} u \tag{1.92}$$

□

Remark 1.8. *For harmonic functions,*

$$\min_{x\in\bar{U}} u = \min_{x\in\partial U} u \tag{1.93}$$

Under suitable assumptions on the domain U, the weak maximum principle can be extended to the strong maximum principle.

Theorem 1.5 (Strong maximum principle)**.** *Let $u \in C^2(U) \cap C(\bar{U})$, harmonic in $U \subset \mathbb{R}^n$ open and connected, and suppose that there exists $x_0 \in U$ such that*

$$u(x_0) = \max_{x\in\bar{U}} u(x). \tag{1.94}$$

Then, u is constant in U.

Proof. Suppose that there exists a point $x_0 \in U$ such that, for some constant $0 < M < \infty$,

$$u(x_0) = \max_{x\in\bar{U}} u(x) \leqslant M. \tag{1.95}$$

Let us call

$$V := \{x \in U \big| u(x) = M\}. \tag{1.96}$$

Let now $0 < r < \text{dist}(x_0, \partial U)$: by the mean value theorem we know that

$$M = u(x_0) = \fint_{B(x_0,r)} u(y)dy \leqslant M \Longrightarrow u(y) = M, \forall y \in B(x_0, r). \tag{1.97}$$

This means that $B(x_0, r) \subset V \Rightarrow$ the set V is open. Let us rewrite:

$$U = V \cup (U\backslash V), \tag{1.98}$$

where $U\backslash V = \big\{x \in U \big| u(x) \neq M\big\}$. By continuity of u, this set is open as well. Clearly $U \cap (U\backslash V) = \varnothing$. Therefore, by definition of connected set,

$$\text{either } V = \{x \in U \big| u(x) = M\} = \varnothing \text{ or } U\backslash V = \big\{x \in U \big| u(x) \neq M\big\} = \varnothing. \tag{1.99}$$

However, by assumption $U \neq \varnothing$, therefore $U\backslash V = \varnothing \Longrightarrow u(x)$ is constant in U. □

Remark 1.9. *The strong maximum principle implies the weak maximum principle, and it can be shown by just applying the strong maximum principle to each connected component in case U is disconnected. Moreover, it generalizes the weak maximum principle because it tells us that the only way in which $u(x)$ can reach its maximum in U is via the constant function.*

Applications of the maximum principle.

Proposition 3. *Let $U \subset \mathbb{R}^n$ be connected and $u \in C^2(U) \cap C(\bar{U})$ such that*

$$\begin{cases} \Delta u = 0, & \text{in } U, \\ u = g, & \text{on } \partial U, \end{cases} \tag{1.100}$$

with $g \geqslant 0$. If $g > 0$ somewhere on ∂U, then $u > 0$ everywhere in U.

Proof. Suppose that there exists $x_0 \in U$ such that $u(x_0) < 0$, and let $\tilde{u} = -u$, $\tilde{g} = -g$. $\tilde{u}$ is harmonic in $U \Rightarrow$

$$\underbrace{\max_{x\in\bar{U}} \tilde{u}}_{\geqslant 0} = \max_{x\in\partial U} \bar{u} = \underbrace{\max_{x\in\partial U} \tilde{g}}_{\leqslant 0}, \tag{1.101}$$

which is a contradiction. □

The strong maximum principle allows to prove the uniqueness of the solution of the initial value problem of the Poisson equation.

Theorem 1.6 (Uniqueness of the solution of the initial value Poisson problem)**.** *Let $g \in C(\partial U)$, $f \in C(U)$, with U open and bounded. Then, there exists at most one solution $u \in C^2(U) \cap C(\bar{U})$ of the initial value problem*

$$\begin{cases} -\Delta u = f & \text{in } U, \\ u = g & \text{on } \partial U. \end{cases} \tag{1.102}$$

Proof. Let us suppose, by contradiction, that there exist two solutions $u, \tilde{u}$, and let us define $h_\pm = \pm(u - \tilde{u})$. Of course, $h_\pm$ solve the initial value problem

$$\begin{cases} \Delta h_\pm = 0 & \text{in } U, \\ h_\pm = 0 \text{ on } \partial U, \end{cases} \tag{1.103}$$

hence it is harmonic in U. Then, the maximum of $\pm(u - \tilde{u})$ is reached on ∂U, which implies

$$u = \tilde{u}. \tag{1.104}$$

□

Remark 1.10. *This theorem does not apply to unbounded domains, e.g. $U = \mathbb{R}^n$. In this case, we know one solution but we do not know whether it is unique. Instead, for U bounded we do not know yet how to prove uniqueness of the solution, in general. In the trivial case $g = const$, the unique solution is $u = g = const$.*

For unbounded domains, uniqueness can only hold in a restricted class of functions. Consider

$$-\Delta u = f \qquad \text{in } \mathbb{R}^n \backslash \{0\}.$$

Then the solution is trivially not unique if we allow u to be unbounded: if u is a solution, then also $u + u_0$ is solution if $\Delta u_0 = 0$. Thus, we can take $u_0 = \Phi =$ fundamental solution of Laplace's equation. In $n \geqslant 2$ dimensions, $\Phi(x) \to \infty$ at $|x| \to 0$, meaning that u is unbounded.

1.3.6 Regularity

Here we will establish some regularity properties of harmonic function. We shall use the notion of *mollifier*, which we briefly recall.

Mollifiers. Let

$$\eta(x) = \begin{cases} ce^{\frac{1}{|x|^2-1}} & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geqslant 1. \end{cases} \tag{1.105}$$

By definition, $\eta \in C^\infty(\mathbb{R}^n)$, and the constant c is chosen in such a way that $\int \eta(x) = 1$. We also define a rescaled version of η:

$$\eta_\epsilon(x) = \frac{1}{\epsilon^n}\eta(\frac{x}{\epsilon}). \tag{1.106}$$

Notice that

$$\mathrm{supp}(\eta) := \overline{\{x \in \mathbb{R}^n \big| \eta(x) \neq 0\}} = \overline{B(0,1)}, \tag{1.107}$$

while $\mathrm{supp}\, \eta_\epsilon = \overline{B(0,\epsilon)}$. Notice also that $\int \eta_\epsilon(x)dx = 1$. We call η_ϵ the *standard mollifier*. One can "smoothen" a given function by taking the convolution with η_ϵ. Let

$$U_\epsilon := \{x \in U \big| \mathrm{dist}(x, \partial U) > \epsilon\}. \tag{1.108}$$

We define

$$u_\epsilon = \eta_\epsilon * u := \int_U dyu(y)\eta_\epsilon(x-y) = \int_{B(0,\epsilon)} \eta_\epsilon(y)u(x-y)dy. \tag{1.109}$$

Remark 1.11. *The set U_ϵ is defined in such a way that $x \in U_\epsilon$, $y \in B(0,\epsilon) \Longrightarrow x - y \in U$.*

Proposition 4. *For small ϵ, u_ϵ is a smooth approximation of u in the following sense:*

1. *$u_\epsilon \in C^\infty(U_\epsilon)$.*
2. *$u_\epsilon \underset{\epsilon \to 0}{\longrightarrow} u$ almost everywhere.*

Proof. 1. Fix $x \in U_\epsilon$, $i \in \{1, \cdot, n\}$ and h so small that $x + he_i \in U_\epsilon$. Then

$$\begin{aligned} \frac{u^\epsilon(x+he_i) - f^\epsilon(x)}{h} &= \frac{1}{\epsilon^n}\int_U \frac{1}{h}\left[\eta\left(\frac{x+he_i-y}{\epsilon}\right) - \eta\left(\frac{x-y}{\epsilon}\right)\right]u(ydy) = \\ &= \frac{1}{\epsilon^n}\int_V \frac{1}{h}\left[\eta\left(\frac{x+he_i-y}{\epsilon}\right) - \eta\left(\frac{x-y}{\epsilon}\right)\right]u(ydy), \end{aligned} \tag{1.110}$$

for some open set $V \subset\subset U$. Since

$$\frac{\eta\left(\frac{x+he_i-y}{\epsilon}\right) - \eta\left(\frac{x-y}{\epsilon}\right)}{h} \underset{h \to 0}{\longrightarrow} \eta_{\epsilon, x_i}(x-y), \tag{1.111}$$

uniformly in V, we can bring the limit in the integral so that

$$\lim_{h \to 0} \frac{u^\epsilon(x+he_i) - f^\epsilon(x)}{h} = \int_U dyu(y)\eta_{\epsilon,x_i}(x-y). \tag{1.112}$$

We can repeat this procedure for all $D^\alpha u_\epsilon$.

2. Suppose that u is continuous. Using that $\int \eta_\epsilon(y)dy = 1$,

$$\begin{aligned}|u_\epsilon(x) - u(x)| &= \left|\int_{B(x,\epsilon)} \eta_\epsilon(x-y)\,(u(y)-u(x))\,dy\right| \leqslant \\ &\leqslant \frac{1}{\epsilon^n}\int_{B(x,\epsilon)} \eta\left(\frac{x-y}{\epsilon}\right)|u(y)-u(x)|dy \leqslant \\ &\leqslant C \fint_{B(x,\epsilon)} |u(y)-u(x)|\,dy \xrightarrow[\epsilon\to 0]{} 0.\end{aligned} \tag{1.113}$$

By using some notions of measure theory, one could prove convergence almost everywhere by only assuming that u is locally integrable.

□

Theorem 1.7 (Smoothness of harmonic functions)**.** *If $u \in C(U)$ satisfies*

$$u(x) = \fint_{\partial B(x,r)} u(y)dS(y), \qquad \forall B(x,r) \subset U, \tag{1.114}$$

then $u \in C^\infty(U)$.

Proof.

$$\begin{aligned}u^\epsilon(x) &= \int_U \eta_\epsilon(x-y)u(y)dy = \frac{1}{\epsilon^n}\int_{B(x,\epsilon)} \eta\left(\frac{|x-y|}{\epsilon}\right)u(y)dy = \\ &= \frac{1}{\epsilon^n}\int_0^\epsilon \eta\left(\frac{r}{\epsilon}\right)\left(\int_{\partial B(x,r)} u dS\right) = \frac{1}{\epsilon^n}u(x)\int_0^\epsilon \eta\left(\frac{\eta}{\epsilon}\right)n\alpha(n)r^{n-1}dr = \\ &= u(x)\int_{B(0,\epsilon)} \eta_\epsilon dy = u(x).\end{aligned} \tag{1.115}$$

This shows that $u \equiv u_\epsilon$ on U_ϵ, so $u \in C^\infty(U_\epsilon)$ for each $\epsilon > 0$. □

Theorem 1.8 (Estimates on the derivatives of harmonic functions)**.** *Let u be harmonic in U. Then*

$$|D^\alpha u(x_0)| \leqslant \frac{C_K}{r^{n+K}}\|u\|_{L^1(B(x_0,r))} \tag{1.116}$$

$\forall B(x_0,r) \subset U$ and $|\alpha| = K$, and where:

$$C_0 = \frac{1}{\alpha(n)}, \qquad C_K = \frac{(2^{n+1}nK)^K}{\alpha(n)} \tag{1.117}$$

Remark 1.12. $\|u\|_{L^1(B(x_0,r))} = \int_{B(x_0,r)} dy|u(y)|$.

Proof. We shall proceed by induction on the order of the derivatives $K = 0, 1, 2, \cdots$. Consider first the case $K = 0$: by the mean value formula we have

$$u(x_0) = \fint_{B(x_0,r)} dyu(y) = \frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} dyu(y) \tag{1.118}$$

so we can bound

$$|u(x_0)| \leqslant \frac{1}{B(x_0,r)} \int_{B(x_0,r)} dy|u(y)| = \frac{1}{r^n \alpha(n)} \|u\|_{L^1(B(x_0,r))}. \tag{1.119}$$

Consider now $K = 1$: we know that u is harmonic, so u_{x_i} is also harmonic, and we can use again the mean value formula

$$u_{x_i}(x_0) = \fint_{B(x_0,r/2)} u_{x_i}(y)dy. \tag{1.120}$$

Therefore, by Gauss-Green:

$$|u_{x_i}(x_0)| \leqslant \frac{2^n}{\alpha(n)r^n} \underbrace{\left| \int_{B(x_0,r/2)} dy u_{x_i} \right|}_{\left|\int_{\partial B(x_0,r/2)} dS(y)\nu_i u(y)\right|} \leqslant \frac{2^n}{\alpha(n)r^n} \sup_{y \in \partial B(x_0,r/2)} |u(y)| \underbrace{|\partial B(x_0,r/2)|}_{n\alpha(n)r^{n-1}2^{-(n-1)}}, \tag{1.121}$$

so that

$$|u_{x_i}(x_0)| \leqslant \frac{2n}{r} \|u\|_{L^\infty(\partial B(x_0,r/2))}. \tag{1.122}$$

Since $B(x_0, r/2) \subset B(x_0, r) \subset U$, we can use the argument we used in the case $K = 0$ to bound the $\|u\|_{L^\infty(\partial B(x_0,r/2))}$, getting

$$\|u\|_{L^\infty(\partial B(x_0,r/2))} \leqslant \frac{1}{\alpha(n)} \left(\frac{2}{r}\right)^n \|u\|_{L^1(B(x_0,r))}, \tag{1.123}$$

and, summarizing,

$$|u_{x_i}(x_0)| \leqslant \frac{2n}{r} \frac{1}{\alpha(n)} \frac{2^n}{r^n} \|u\|_{L^1(B(x_0,r))} = \frac{2^{n+1}n}{\alpha(n)r^{n+1}} \|u\|_{L^1(B(x_0,r))} \tag{1.124}$$

which proves the claim in the case $K = 1$.
Generic step: suppose that (1.116) holds $\forall |\alpha| \leqslant K-1$. Then consider the next step $|\alpha| = K$. We have $D^\alpha = \left(D^\beta u\right)_{x_i}$ for some multi-index β such that $|\beta| = K - 1$, therefore for $B(x_0, r/K) \subset U$:

$$|D^\alpha u(x_0)| \leqslant \frac{nK}{r} \|D^\beta u\|_{L^\infty(\partial B(x_0,r/K))}, \tag{1.125}$$

analogously to what we had before. So, if $x \in \partial B(x_0, r/K)$ then $B(x, \frac{K-1}{K}r) \subset B(x_0, r) \subset U$, hence (1.116) for $K - 1$ implies

$$|D^\beta u| \leqslant \frac{(2^{n+1}n(K-1))^{K-1}}{\alpha(n)\left(\frac{K-1}{K}r\right)^{n+K-1}} \|u\|_{L^1(B(x_0,r))}. \tag{1.126}$$

Plugging this estimate in the previous one, we get

$$|D^\alpha u(x_0)| \leqslant \frac{(2^{n+1}nK)^K}{\alpha(n)r^{n+K}} \|u\|_{L^1(B(x_0,r))}. \tag{1.127}$$

□

Remark 1.13. *The main point of the proof is to find the right sequence of constants C_K. These estimates are analogous to the bounds for the derivatives of analytic functions on $\mathbb{C}$, that can be derived from $f(z) := \frac{1}{2\pi i}\int_{\mathcal{C}} d\omega \frac{f(\omega)}{\omega - z}$ (recall (1.5)).*

Corollary 1 (Liouville's Theorem)**.** *Let $u : \mathbb{R}^n \to \mathbb{R}$ harmonic, and suppose that u is bounded. Then, u is constant.*

Proof.

$$|Du(x_0)| \leqslant \frac{C}{r^{n+1}}\|u\|_{L^1(B(x_0,r))} \leqslant \frac{C}{r^{n+1}}|B(x_0,r)| \leqslant \frac{C}{r} \xrightarrow[r\to\infty]{} 0. \tag{1.128}$$

Therefore, $Du = 0$ and hence u is constant. □

Corollary 2 (Solution of Poisson's initial value problem in $\mathbb{R}^n$)**.** *Let $f \in C_c^2(\mathbb{R}^n)$, $n \geqslant 3$. Then, any bounded solution of $-\Delta u = f$ in $\mathbb{R}^n$ has the form*

$$u(x) = \int_{\mathbb{R}^n} \phi(x-y) f(y) dy + C \tag{1.129}$$

for some constant C.

Proof. Looking at the explicit form of u, we have

$$|u(x)| \leqslant \left| \int dy \phi(x-y) f(y) \right| \leqslant C \left| \int dy \phi(x-y) \right| \leqslant \infty, \tag{1.130}$$

so u is bounded. Suppose now that $\tilde{u}$ is another solution of $-\Delta\tilde{u} = f$ on $\mathbb{R}^n$. Then, $-\Delta(u-\tilde{u}) = 0$, but since $u - \tilde{u}$ is bounded, $u - \tilde{u} = \text{const}$ by Liouville's theorem. □

After these important corollaries, we come back to remark 1.13 and prove the following theorem.

Theorem 1.9. *Let u be harmonic in U. Then u, is analytic in U.*

Proof. We have to show that, for any $x_0 \in U$, there exists a neighbourhood of x_0 such that u can be represented as a convergent power series in $x - x_0$. Let us call

$$r := \frac{1}{4}\text{dist}(x_0, \partial U), \qquad M := \frac{1}{\alpha(n) r^n}\|u\|_{L^1(B(x_0,2r))} < \infty. \tag{1.131}$$

For each $x \in B(x_0, r)$, by construction we have that $B(x,r) \subset B(x_0, 2r)$, and we already got the bound

$$\|D^\alpha u\|_{L^\infty(B(x_0,r))} \leqslant M \left(\frac{2^{n+1} n}{r}\right)^{|\alpha|} |\alpha|^{|\alpha|}. \tag{1.132}$$

Using the Stirling formula

$$\lim_{k\to\infty} \frac{k^{k+\frac{1}{2}}}{k! e^k} = \frac{1}{(2\pi)^{\frac{1}{2}}}, \tag{1.133}$$

we get

$$|\alpha|^{|\alpha|} \leqslant C e^{|\alpha|} |\alpha|! \tag{1.134}$$

for some constant C and all multi-indices α. Invoking the multinomial theorem:

$$n^K = (1 + \cdots + 1)^k = \sum_{|\alpha|=k} \frac{|\alpha|!}{\alpha!}, \tag{1.135}$$

from which we get

$$|\alpha|! \leqslant n^{|\alpha|}\alpha!. \tag{1.136}$$

Combining these inequalities we get

$$\|D^\alpha u\|_{L^\infty(B(x_0,r))} \leqslant CM \left(\frac{2^{n+1}n^2 e}{r}\right)^{|\alpha|} \alpha!. \tag{1.137}$$

The Taylor series for u at x_0 is

$$\sum_\alpha \frac{D^\alpha u(x_0)}{\alpha!}(x - x_0)^\alpha, \tag{1.138}$$

where the sum is taken over all multi-indices. The claim is that this power series converges provided

$$|x - x_0| < \frac{r}{2^{n+2}n^3 e}. \tag{1.139}$$

In order to check this claim, we have to compute, for each $N \in \mathbb{N}$, the remainder

$$R_N(x) := u(x) - \sum_{k=0}^{N-1} \sum_{|\alpha|=k} \frac{D^\alpha u(x_0)(x - x_0)^\alpha}{\alpha!} = \sum_{|\alpha|=N} \frac{D^\alpha u(x_0 + t(x - x_0))(x - x_0)^\alpha}{\alpha!}, \tag{1.140}$$

for some t depending on x, $0 \leqslant t \leqslant 1$. Finally, using the estimates on the derivatives we get

$$|R_N(x)| \leqslant CM \sum_{|\alpha|=N} \left(\frac{2^{n+1}n^2 e}{r}\right)^N \left(\frac{r}{2^{n+2}n^3 e}\right)^N \leqslant CMn^N \frac{1}{(2n)^N} = \frac{CM}{2^N} \underset{N\to\infty}{\to} 0. \tag{1.141}$$

□

The next theorem allows to show that maxima and minima of harmonic functions are in some sense comparable.

Theorem 1.10 (Harnack inequality)**.** *Let U be a bounded open subset of $\mathbb{R}^n$. Let $V \subset\subset U$ connected and open, and u harmonic in U such that $u \geqslant 0$. Then, there exists a constant $C > 0$ such that*

$$\sup_V u \leqslant C \inf_V u. \tag{1.142}$$

Moreover, since

$$\inf_V u \leqslant u(y) \qquad \forall y \in V, \qquad \sup_V u \geqslant u(y) \qquad \forall y \in V \tag{1.143}$$

then

$$u(x) \leqslant Cu(y) \qquad \forall x, y \in V \tag{1.144}$$

that is:

$$\frac{1}{C}u(y) \leqslant u(x) \leqslant Cu(y). \tag{1.145}$$

Proof. Let $r := \frac{1}{4}\text{distt}(V, \partial U)$. If $x, y \in V$, $|x - y| \leqslant r$, then

$$u(x) = \fint_{B(x,2r)} u(x)dz \geqslant \frac{1}{|B(x,2r)|} \int_{B(y,r)} u(z)dz \tag{1.146}$$

being $u \geqslant 0$. In particular

$$\frac{1}{|B(x,2r)|} \int_{B(y,r)} u(z)dz = \frac{1}{2^n} \fint_{B(y,r)} u(z)dz = \frac{1}{2^n} u(y) \tag{1.147}$$

so that

$$2^n u(y) \geqslant u(x) \geqslant \frac{1}{2^n} u(y) \qquad \forall x, y \in V, |x - y| \leqslant r. \tag{1.148}$$

Now, since V is connected and $\bar{V}$ is compact, we can cover $\bar{V}$ with finitely many balls $\{B_i\}_{i=1}^N$ of radius $r/2$ and $B_i \cap B_{i-1} \neq \varnothing$. This implied that

$$u(x) \geqslant \frac{1}{2^{n(N+1)}} u(z) \qquad \forall x, z \in V, \tag{1.149}$$

where $N = N_V$ depends on V. □

Properties of the solutions of Poisson's initial value problem. We now have enough tools to discuss the simplest extension of Newton's law, in the case f is not radial.

Example 1.3.1 (Multipole expansion)**.** Let $\Phi(x)$ be the fundamental solution of Laplace's equation. Consider

$$u = \int dy f(y) \Phi(x - y), \qquad x \neq U \supset \text{supp} f. \tag{1.150}$$

Being $\Phi(x - y)$ harmonic, it is also analytic, so we can expand around $y = 0$:

$$\Phi(x - y) = \Phi(x) - y(D_y\Phi)(x) + \frac{1}{2} y_i y_j \partial_{y_i} \partial_{y_j} \Phi(x) + \cdots, \tag{1.151}$$

and in particular

$$u(x) = \int dy f(y) \Phi(x - y) = \underbrace{\int dy f(y)}_{a} \Phi(x) - \int dy f(y) y_i \partial_i \Phi(x) + \mathcal{E}(x). \tag{1.152}$$

In the case $n \geqslant 3$, $\Phi(x) = \frac{1}{4\pi|x|}$ and $\partial_i \Phi(x) = \frac{-x_i}{4\pi|x|^3}$, so that

$$u(x) = \underbrace{\frac{a}{4\pi|x|}}_{1-\text{charge potential}} + \underbrace{\frac{p \cdot x}{4\pi|x|^3}}_{\text{dipole momentum}} + \mathcal{O}\left(\frac{1}{|x|^4}\right) \tag{1.153}$$

which can be interpreted as a potential generated by to point-like charges sitting at x_1 and x_2, with charges respectively $\pm q$ such that $p = q_1 x_1 + q_2 x_2 = q(x_1 - x_2)$.

1.3.7 Green's function

We want now to discuss the existence of the solution of the Poisson initial value problem (1.100), that we recall here for simplicity

$$\begin{cases} -\Delta u = f, & \text{in } U, \\ u = g, & \text{on } \partial U. \end{cases}$$

with prescribed f and g. We know (Thm. (1.6)) that this initial value problem has at most one solution. As we we shall see, existence can be proved in certain cases. In particular, we will discuss a useful representation formula for the solution of the initial value problem that allows to write an explicit expression for the solution u, whenever it exists. This expression can actually be used to prove the existence of the solution in some simple cases.

Definition 10 (Green's first identity)**.** *Let $U \subset \mathbb{R}^n$, and $u, v \in C^2(\bar{U})$. Then:*

$$\int_U v\Delta u dx + \int_U Du \cdot Dv dx = \int_{\partial U} v\nu \cdot D_y u dS(y). \tag{1.154}$$

By just interchanging the rule of u, v and subtracting the two identities, we get the second Green's identity.

Definition 11 (Green's second identity)**.** *Let $U \subset \mathbb{R}^n$, and $u, v \in C^2(\bar{U})$. Then:*

$$\int_U (v\Delta u - u\Delta u) = \int_{\partial U} (v\nu \cdot D_y u - u\nu \cdot D_y v) dS(y) \tag{1.155}$$

In particular, we would like to apply such identities to the case in which u is a solution of the Poisson initial value problem (1.100) and v is the fundamental solution of Laplace's equation:

$$v(y) = \begin{cases} \frac{1}{2\pi} \log|x-y| & \text{if } n = 2, \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x-y|^{n-2}} & \text{if } n \geqslant 3. \end{cases} \tag{1.156}$$

A priori, the problem in doing that is, as we already commented in (1.43), $\Delta\Phi$ is not integrable due to the singularity at $x = y$. To solve this problem, let us just start by considering the domain $U \backslash B(x, \epsilon)$:

$$\begin{aligned} &\int_{U\backslash B(x,\epsilon)} (\Phi(x-y)\Delta u - u\Delta\Phi(x-y))\, dy = \\ &= \int_{\partial(U\backslash B(x,\epsilon))} (\Phi(x-y)\nu \cdot D_y u(y) - u(y)\nu \cdot D_y\Phi(x-y))\, dS(y). \end{aligned} \tag{1.157}$$

Now we use that $\Delta_y\Phi(x-y) = 0$ for each $y \in U\backslash B(x_0, \epsilon)$, and also that the integral appearing in the r.h.s. can be rewritten as the sum of the integral along two boundaries:

$$\begin{aligned} &\int_{\partial_{(U\backslash B(x,\epsilon))}} (\Phi(x-y)\nu \cdot D_y u(y) - u(y)\nu \cdot D_y\Phi(x-y))\, dS(y) = \\ &= \underbrace{\int_{\partial U} (\Phi(x-y)\nu \cdot D_y u(y) - u(y)\nu \cdot D_y\Phi(x-y))\, dS(y)}_{\text{with outward normal}} + \\ &+ \underbrace{\int_{\partial B(x,\epsilon)} (\Phi(x-y)\nu \cdot D_y u(y) - u(y)\nu \cdot D_y\Phi(x-y))\, dS(y)}_{\text{with inward normal}}. \end{aligned} \tag{1.158}$$

Hence,

$$\begin{aligned}\int_{\partial(U\setminus B(x,\epsilon))}\Phi(x-y)\Delta u(y)dy &= \int_{\partial U}\left(\Phi(x-y)\nu\cdot D_y u(y) - u(y)\nu\cdot D_y\Phi(x-y)\right)dS(y)+ \\ &\quad + \int_{\partial B(x,\epsilon)}\left(\Phi(x-y)\nu\cdot D_y u(y) - u(y)\nu\cdot D_y\Phi(x-y)\right)dS(y).\end{aligned} \tag{1.159}$$

We would like to take the $\epsilon\to 0$ limit: with this purpose, notice that

$$\begin{aligned}&\left|\int_{\partial B(x,\epsilon)}\Phi(x-y)\nu\cdot D_y u(y)dS(y)\right| \leqslant |\Phi(\epsilon)|\int_{\partial B(x,\epsilon)}|D_y u(y)|\,dS(y)\leqslant \\ &\leqslant |\Phi(\epsilon)|\sup_{y\in B(x,\epsilon)}|D_y u(y)|\int_{\partial B(x_\epsilon)}dS(y)\leqslant C\epsilon^{n-1}|\Phi(\epsilon)|\xrightarrow[\epsilon\to 0]{} 0.\end{aligned} \tag{1.160}$$

Now we have to bound the other term:

$$\begin{aligned}&\int_{\partial B(x,\epsilon)}u(y)\nu\cdot D_y\Phi(x-y) = \int_{\partial B(x,\epsilon)}u(y)\nu\cdot\Phi'(x-y)\frac{x-y}{|x-y|} = \\ &= -\int_{\partial B(x,\epsilon)}u(y)\Phi'(|x-y|)\frac{|x-y|^2}{|x-y|^2} = -\Phi'(\epsilon)\int_{\partial B((x,\epsilon)}u(y)dS(y) = \\ &= -\frac{1}{n\alpha(n)\epsilon^{n-1}}\epsilon^{n-1}n\alpha(n)\cdot ⨍_{\partial B(x,\epsilon)}u(y)dS(y)\xrightarrow[\epsilon\to 0^+]{} u(x),\end{aligned} \tag{1.161}$$

where we just used, to pass from the first to the second line, that $\nu(y) = -\frac{x-y}{|x-y|}$. All in all,

$$\begin{aligned}&\lim_{\epsilon\to 0}\int_{U\setminus B(x,\epsilon)}\Phi(x-y)\Delta u(y)dy = \\ &= \int_{\partial U}\left[\Phi(x-y)\nu\cdot D_y u(y) - u(y)\nu\cdot D_y\Phi(x-y)\right]dS(y) + u(x),\end{aligned} \tag{1.162}$$

and, since

$$\lim_{\epsilon\to 0}\int_{U\setminus B(x,\epsilon)}\Phi(x-y)\Delta u(y)dy = \int_U\Phi(x-y)\Delta u(y)dy, \tag{1.163}$$

we finally get

$$u(x) = \int_U\Phi(x-y)\Delta u(y)dy + \int_{\partial U}\left[-\Phi(x-y)\nu\cdot D_y u(u) + u(y)\nu\cdot D_y\Phi(x-y)\right]dS(y)\,. \tag{1.164}$$

Remark 1.14. *The last equation (1.164) tells us that to find $u(x)$ in U it is enough to know $\partial_y u(y)$ on the boundary ∂U.*

Consider now the solution of Laplace's equation in $U\subset\mathbb{R}^n$: $\phi^x(y)$ such that $\Delta_y\phi(y) = 0$ in U. By Green's second identity we get

$$\int_U\phi^x\Delta u dy = \int_{\partial U}\left(\phi^x\nu\cdot D_y u - u\nu\cdot D_y\phi^x\right)dS(y), \tag{1.165}$$

so that we can just add this identity to (1.164) that we get before, obtaining

$$
\begin{aligned}
u(x) &= \int_U (\Phi(x-y) + \phi^x(y))\Delta u dy + \\
&+ \int_{\partial U} \left[(-\Phi(x-y) - \phi^x(y))\, \nu \cdot D_y u(y) + u(y)\nu \cdot D_y \Phi(x-y) + \phi^x(y) \right] dS(y) =: \\
&=: \int_U G\Delta u dy + \int_{\partial U} \left[-G\nu \cdot D_y u(y) + u(y)\nu \cdot D_y G \right] dS(y)
\end{aligned} \tag{1.166}
$$

where we defined

$$G(x,y) = \Phi(x-y) + \phi^x(y). \tag{1.167}$$

Remark 1.15. *Suppose now that we can choose $\phi^x(y)$ such that $G = 0$ on ∂U. Then*

$$u(x) = \int_U G\Delta u(y)dy + \int_{\partial U} u(y)\nu \cdot D_y G dS(y), \tag{1.168}$$

meaning that, if u is a solution of the initial value problem

$$-\Delta u = f \text{ in } U, u = g \text{ on } \partial U,$$

then (1.164) gives an explicit expression for u as a function of f and g, which are the prescribed functions of the problem. Conversely, if one can prove that the Laplace initial value problem

$$\Delta \phi^x = 0 \text{ in } U, \phi^x = \Phi(x-y) \text{ on } \partial U,$$

admits a solution, then (1.164) gives a solution for the Poisson initial value problem (1.100).

Definition 12. *$G \equiv G(x,y)$ is called Green's function for $U \subset \mathbb{R}^n$ for $x, y \in U, x \neq y$.*

Before discussing the computation of the Green's function is some special cases, let us discuss some symmetry property of the Green's function.

Theorem 1.11 (Symmetry of the Green function)**.** $\forall x, y \in U, x \neq y$,

$$G(x,y) = G(y,x). \tag{1.169}$$

Proof. Let $v(z) = G(x,z)$, $w(z) = G(y,z)$. Then by construction $\Delta v = 0$ if $z \neq x$, and $\Delta w = 0$ if $z \neq y$, and $v = w = 0$ on ∂U. Let now

$$V := U \backslash B(x,\epsilon) \cup B(y,\epsilon), \qquad \epsilon \text{ small}. \tag{1.170}$$

Using Green's identity,

$$\int_{\partial B(x,\epsilon)} (\nu \cdot D_z v w - \nu \cdot D_z w v)\, dS(z) = \int_{\partial B(y,\epsilon)} (\nu \cdot D_z w v - \nu \cdot D_z v w)\, dS(z). \tag{1.171}$$

Being w smooth close to x,

$$\left| \int_{\partial B(x,\epsilon)} \nu \cdot D_z w v dS(z) \right| \underset{\epsilon \to 0}{\longrightarrow} 0. \tag{1.172}$$

Also, we can conclude that

$$\int_{\partial B(x,\epsilon)} \nu \cdot D_z v w = \int_{\partial B(x,\epsilon)} \nu \cdot D_z \Phi(x-z) w(z) \underset{\epsilon \to 0}{\longrightarrow} w(z). \tag{1.173}$$

Similarly, the r.h.s. of

$$\int_{\partial B(y,\epsilon)} (\nu \cdot D_z w v - \nu \cdot D_z v w)\, dS(z) \underset{\epsilon \to 0}{\longrightarrow} v(y). \tag{1.174}$$

Therefore $w(z) = G(y,x) = v(y) = G(x,y)$. □

Example 1.3.2 (Green's function in the half space)**.** Let us consider the half space $\mathbb{R}^n_+ := \{x = (x_1, \cdot, x_n), | x_n > 0\}$.

Notice that this region is unbounded, so we cannot rely on the previous results. We shall compute the Green function using a reflection trick, known as *method of the images*.

Definition 13 (Reflection)**.** *Given $x \in \mathbb{R}^n_+$, we define its reflection with respect to $\partial\mathbb{R}^n_+ = \{x = (x_1, \cdot, x_n), | x_n = 0\}$ as*

$$\tilde{x} = (x_1, \cdots, x_{n-1}, -x_n). \tag{1.175}$$

We are looking for a function $\phi^x(y)$ such that

$$\begin{cases} \Delta_y \phi^x(y) = 0 & \text{in } \mathbb{R}^n_+, \\ \phi^x(y) = \Phi(x-y) & \text{on } \partial\mathbb{R}^n_+. \end{cases} \tag{1.176}$$

The problem is that $\phi^x(y)$ is not defined at $x = y$. Therefore consider the function

$$\phi^x(y) = \Phi(\tilde{x} - y). \tag{1.177}$$

If $x \in \mathbb{R}^n_+ \Rightarrow \tilde{x} \notin \mathbb{R}^n_+$, therefore $\phi^x(y)$ is well defined $\forall x, y \in \mathbb{R}^n_+$ and, in particular

$$\begin{cases} \Delta_y \phi^x(y) = 0 & \text{in } \mathbb{R}^n_+, \\ \phi^x(y) = \Phi(x-y) & \text{on } \partial\mathbb{R}^n_+ \end{cases} \tag{1.178}$$

being $x = \tilde{x}$ on $\partial\mathbb{R}^n_+$. So the Green function for the half plane $\mathbb{R}^n_+$ is

$$G(x,y) = \Phi(x-y) - \Phi(\tilde{x} - y), \qquad x, y \in \mathbb{R}^n_+, x \neq y. \tag{1.179}$$

Let us now use G to compute the solution of the Poisson initial value problem (1.100). We have

$$G_{y_n}(x,y) = \Phi_{y_n}(x-y) - \Phi_{y_n}(\tilde{x}-y) = \frac{1}{n\alpha(n)} \left[\frac{y_n - x_n}{|y-x|^n} - \frac{y_n + x_n}{|y - \tilde{x}|^n} \right]. \tag{1.180}$$

Recalling that

$$u(x) = \int_U G \Delta u + \int_{\partial U} u(y) \nu \cdot D_y G, \tag{1.181}$$

for $y \in \partial\mathbb{R}^n_+$

$$\nu \cdot D_y G(x,y) = -G_{y_n}(x,y) = \frac{2x_n}{n\alpha(n)} \frac{1}{|x-y|^n}. \tag{1.182}$$

Suppose that u solves the initial value problem

$$\begin{cases} \Delta u = 0, & \text{in } \mathbb{R}^n_+, \\ u = g, & \text{on } \partial\mathbb{R}^n_+, \end{cases} \tag{1.183}$$

we expect that, for $x \in \mathbb{R}^n_+$,

$$u(x) = \frac{2x_n}{n\alpha(n)} \int_{\partial\mathbb{R}^n} \frac{g(y)}{|x-y|^n} dy = \int \mathcal{K}(x,y)g(y)dy, \tag{1.184}$$

where we have introduced the *Poisson kernel* $\mathcal{K}$ for the half plane. To conclude, we are left with proving that u is indeed a solution of (1.183).

Proposition 5. *Let $g \in C(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$, hence*

1. *$u \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n_+)$,*
2. *$\Delta u = 0$ in $\mathbb{R}^n_+$,*
3. *for $x \in \mathbb{R}^n_+$, $\lim_{x\to\bar{x}} u(x) = g(\bar{x})$ for each $\bar{x} \in \partial\mathbb{R}^n_+$.*

Proof. 1. Since G is harmonic for $x \neq y$, $\mathcal{K}(x,y) = -G_{y_n}(x,y)$ is also harmonic for $x \neq y$, so it is harmonic for $x \in \mathbb{R}^n_+, y \in \partial\mathbb{R}^n_+$.

2. $\Delta u(x) = \Delta_x \int_{\partial\mathbb{R}^n_+} dy \mathcal{K}(x,y)g(y) = \int dy \Delta_x \mathcal{K}(x,y)g(y) = 0$.

3.

$$\begin{aligned} |u(x) - g(\bar{x})| \leqslant &\int_{\partial\mathbb{R}^n_+} \mathcal{K}(x,y)|g(y) - g(\bar{x})| = \\ = &\int_{\partial\mathbb{R}^n_+ \cap B(\bar{x},j)} \mathcal{K}(x,y)|g(y) - g(\bar{x})| + \int_{\partial\mathbb{R}^n_+ \setminus B(\bar{x},j)} \mathcal{K}(x,y)|g(y) - g(\bar{x})| \leqslant \\ \leqslant &C\epsilon + \int_{\partial\mathbb{R}^n_+ \setminus B(\bar{x},j)} \mathcal{K}(x,y)|g(y) - g(\bar{x})| \leqslant \\ \leqslant &C\epsilon + c\|g\|_\infty \int_{\partial\mathbb{R}^n_+ \setminus B(\bar{x},j)} \underbrace{\mathcal{K}(x,y)}_{\substack{\simeq \frac{x_n}{|x-y|^n}, \\ \text{integrable in } \partial\mathbb{R}^n_+}} \xrightarrow[x_n \to 0]{} 0. \end{aligned} \tag{1.185}$$

□

1.3.8 Existence of solutions for the Laplace problem

We want to prove existence of solutions for:

$$\begin{cases} \Delta u = 0 & \text{in } U, \\ u = g & \text{on } \partial U, \end{cases} \tag{1.186}$$

where $U \subset \mathbb{R}^n$ is open and bounded and $g \in C^0(\partial U)$.

Definition 14 (Mean-value subharmonic/super-harmonic/harmonic). $u \in C^0(U)$ *is called* mean-value subharmonic *if*

$$u(x) \leqslant \fint_{B(x,r)} u(y)dy, \qquad \forall B(x,r) \subset U. \tag{1.187}$$

$u \in C^0(U)$ *is called* mean-value super-harmonic *if* $-u$ *is mean-value subharmonic, and* mean-value harmonic *if* u *is both mean-value sub-harmonic and mean-value super-harmonic.*

Remark 1.16. *1. If $u \in C^2(U)$, the three situations just describe coincide respectively with*

$$-\Delta u \leqslant 0, \qquad -\Delta u \geqslant 0, \qquad \Delta u = 0. \tag{1.188}$$

2. For harmonic functions, "mean-value harmonic ≡ harmonic". From now on, we will often drop the term "mean-value".

Theorem 1.12.

Let $u \in C^0(U)$. The following statements are equivalent:

1. $u(x) \leqslant \fint_{\partial B(x,r)} u(y)dy \qquad \forall B(x,r) \subset U,$
2. $u(x) \leqslant \fint_{B(x,r)} u(y)dy \qquad \forall B(x,r) \subset U,$
3. $u(x) \leqslant h(x)$, $\forall$ harmonic function h such that, for all $B(x,r) \subset U$, $u \restriction_{\partial B(x,r)} \leqslant h \restriction_{\partial B(x,r)}$.

Proof.

1. $\Rightarrow$ 2. already proven.

2. $\Rightarrow$ 3. Let h be harmonic. Thus, $u - h$ is mean-value subharmonic and $u - h \leqslant 0$ on $\partial B(x,r)$ for all $B(x,r) \subset U$. Hence, by the maximum principle,

$$\sup_{y \in B(x,r)} (u(y) - h(y)) = \sup_{y \in \partial B(x,r)} (u(y) - h(y)) \leqslant 0. \tag{1.189}$$

3. $\Rightarrow$ 1. by the mean value formula. □

Theorem 1.13. *1. Let $u, v \in C^0(U)$, u subharmonic and v super-harmonic. If $u \leqslant v$ on ∂U then either $u < v$ or $u = v$ in U.*

2. Let $u \in C^0(U)$ subharmonic and $B(x,r) \subset U$. We define the harmonic lifting *of u on $B(x,r)$ as*

$$v(y) = \begin{cases} u(y), & \forall y \in U \backslash B(x,r), \\ \int_{\partial B(x,r)} \mathcal{K}_{B(x,r)}(y,z)u(z)dS(z) & \forall y \in B(x,r), \end{cases} \tag{1.190}$$

where $K_{B(x,r)}(y,z)$ is the Poisson Kernel for the ball

$$K_{B(x,r)}(y,z) = \frac{r^2 - |y-x|^2}{n\alpha(n)r} \frac{1}{|y-z|^n}. \tag{1.191}$$

Then, $v \in C^0(U)$, v is subharmonic and $u \leqslant v$.

3. Let $u_1, \cdots, u_k \in C^0(U)$ be subharmonic. Then, $u = \max_{i=,1\cdots,k} u_i$ is subharmonic.

Proof. 1. Follows directly from the maximum principle: $u - v$ is subharmonic and $u - v \leqslant 0$ on ∂U.

2. Let $B' = B(x', r') \subset U$. Let h be harmonic in B' and such that $v \leqslant h$ on $\partial B'$. *Goal: prove that $v \leqslant h$ on B' and that $u \leqslant v$.* Let us start by proving the second statement: by the definition of v (1.190), $u \leqslant v$ for $y \in U \subset B(x,r)$. Let now $y \in B(x,r)$: for all $y \in \partial B(x,r)$ $v(y) = u(y)$ by definition of Poisson kernel. Also, $u - v$ is subharmonic $\Rightarrow u \leqslant v$ $\forall y \in B(x,r)$ by the maximum principle. Therefore, we are left with proving that v is subharmonic in U, and we want to use point 3. of Theorem (1.3.8). Since $u \leqslant v$ in B', so $u \leqslant h$ in B' and

$$u \restriction_{\partial B'} \leqslant v \restriction_{\partial B'} \leqslant h \restriction_{\partial B'} \tag{1.192}$$

$\Rightarrow u \leqslant h$ in B' by the maximum principle. Since $u = v$ on $B' \backslash B$, $v \leqslant h$ in $B' \backslash B$. Suppose now that v is in $B' \cap B$: then v is harmonic. Then, since $v \leqslant h$ on $\partial (B' \cap B)$ we get $v \leqslant g$ in $B' \cap B \subset \partial B' \cup \partial(B' \backslash B) \Rightarrow v \leqslant h$ in B', so v is subharmonic in B' and hence in U.

3. Let $u(y) = \max_i u_i$, and $y \in B(x,r) \subset U$. Let $p \in \{1, \cdots, k\}$ such that

$$u(x) = u_p(x), \qquad x \text{ fixed }. \tag{1.193}$$

So

$$u(x) = u_p(x) \leqslant \fint_{\partial B(x,r)} u_p(y) dS(y) \leqslant \fint_{\partial B(x,r)} u(y) dS(y), \tag{1.194}$$

by definition of u and using that u_p is subharmonic.

□

Finally, we will need some result about the limits of subharmonic functions.

Theorem 1.14. *Let $\{v_k\}_{k \in \mathbb{N}}$ be a nondecreasing sequence of bounded harmonic functions in $U \subset \mathbb{R}^n$ open, bounded and connected. Suppose that*

$$v_k(u) \underset{k \to \infty}{\longrightarrow} u(y) \,.$$

Then,

$$v_k \to u,$$

uniformly in U and u is harmonic.

Proof. Let $z \in U$. The function $v_k(z) - v_\ell(z)$ is harmonic for $k, \ell \in \mathbb{N}$. Moreover, $v_k(z) - v_\ell(z) \geqslant 0$ for $k \geqslant \ell$. By Harnack's inequality

$$\frac{1}{C} \left(v_k(y) - v_\ell(y) \right) \leqslant \left(v_k(z) - v_\ell(z) \right) \leqslant C \left(v_k(y) - v_\ell(y) \right), \tag{1.195}$$

meaning that $(v_k(z) - v_\ell(z))$ is a Cauchy sequence, uniformly in $z \in U$. Let us denote by $u(z)$ the limit of the sequence, and let $B(z,r) \subset U$. By the mean value theorem

$$v_k(z) = \fint_{B(z,r)} v_k(y) dy \underset{k \to \infty}{\longrightarrow} u(z) = \fint_{B(z,r)} u(y) dy, \tag{1.196}$$

which proves that the limit is harmonic. □

Definition 15 (Set of subharmonic functions relative to a continuous function). *Let $g \in C^0(\partial U)$. We define the set of subharmonic functions relative to g as*

$$S_g := \left\{ v \in C^0(\bar{U}) \middle| v \text{ subharmonic and } v \leqslant g \text{ on } \partial U \right\}. \tag{1.197}$$

Theorem 1.15. *Let $g \in C^0(\partial U)$. Define, $\forall x \in \bar{U}$,*

$$u(x) := \sup\{v(x) \big| v \in S_g\}. \tag{1.198}$$

Then, u is well defined and harmonic.

Proof. To begin, notice that $S_g \neq \varnothing$, since the constant function $v(x) = \min_{y \in \partial U} g(y)$ belongs to S_g. Also, notice that the set S_g is bounded: $v(x) \leqslant \max_{y \in \partial U} g(y) < \infty$, $\forall v \in S_g$, which implies that u is well defined. Fix $y \in U$. We denote by $\{v_k(y)\}_{k \in \mathbb{N}}$, $v_k \in S_g$, the sequence such that $\lim_{k \to \infty} v_k(y) = u(y)$.

Assume that $v_k(y) \leqslant v_{k+1}(y)$ (if not, we reorder the $v_k's$ so that this is true). Let us introduce the harmonic lifting

$$w_k(z) = \begin{cases} v_k(z), & \forall y \in U \backslash B(x,r), \\ \int_{\partial B(x,r)} \mathcal{K}_{B(x,r)}(z,t) v_k(t) dS(t) & \forall z \in B(y,r). \end{cases} \tag{1.199}$$

Thanks to Theorem (1.13) we know that w_k is subharmonic and that $v_k \leqslant w_k$. Also, $w_k \in S_g$, since

$$w_k \restriction_{\partial U} = v_k \restriction_{\partial U} \leqslant g_k \restriction_{\partial U}, \tag{1.200}$$

which implies $v_k \leqslant w_k \leqslant u$. Therefore, $v_k(y) \to u(y) \Rightarrow w_k(y) \to u(y)$. Now, thanks to theorem (1.14), using that w_k is harmonic in $B(y,r)$ (up to a rearrangement of the sequence):

$$w_k \to w_*, \tag{1.201}$$

uniformly in $B(y,r)$, with w_* harmonic. We claim that $w_* = u$. We shall proceed by contradiction.

Suppose that $w_* \neq u$ in $B(y,r)$. Therefore, there exists $p \in B(y,r)$ such that $w_*(p) < u(p)$. This also implies that there exists $\tilde{w} \in S_q$ such that $w_*(p) < \tilde{w}(p) < u(p)$. Let $\hat{v}_k = \max\{\tilde{w}, w_k\}$ and

$$\hat{w}_k(z) = \begin{cases} \hat{v}_k(z), & \forall y \in U \backslash B(x,r), \\ \int_{\partial B(x,r)} \mathcal{K}_{B(x,r)}(z,t) \hat{v}_k(t) dS(t) & \forall z \in B(y,r). \end{cases} \tag{1.202}$$

As before, $w_k \leqslant \hat{v}_k \leqslant \hat{w}_k \leqslant u$ and $\hat{w}_k \to \hat{w}_*$ uniformly in $B(y,r)$, $\hat{w}_*$ harmonic. Moreover, $w_k \leqslant \hat{w}_k \Rightarrow w_* \leqslant \hat{w}_*$ and $w_*(y) = \hat{w}_*(y) = u(y)$. Consider now the function $w_* - \hat{w}_*$. By what we proved, we know that: $w_* - \hat{w}_*$ is harmonic; it is $\leqslant 0$; and it reaches its maximum, which is 0, in $y \in B(y,r)$. Therefore, by the strong maximum principle, $w_* = \hat{w}_*$ in $B(y,r)$; that is, $\tilde{w}(p) \leqslant \hat{w}_*(p) = w_*(p)$, which is a contradiction. Hence $w_* = u$, which proves that u is harmonic. ☐

Remark 1.17. *The function u is our candidate for the solution of the Laplace initial value problem (1.186). We are left with checking that the solution fulfills the boundary condition $u = g$ on ∂U.*

Definition 16 (Barrier function, regular point). *Let $U \subset \mathbb{R}^n$ be open, $x_* \in \partial U$. A function $w \in C^0(\bar{U})$ is called a barrier in x_* if:*

1. $w(x_*) = 0$ *and* $w > 0$ *on* $\bar{U}\backslash\{x_*\}$.

2. w *is mean-value super-harmonic in* U.

A point $x_* \in U$ *is called regular if there exist a barrier at* x_*.

Definition 17. *An open set* $U \subset \mathbb{R}^n$ *has the exterior sphere property at* $x_* \in \partial U$ *if there exists* $y \in \mathbb{R}^n$ *and* $r > 0$ *such that*

$$\overline{B(y,r)} \cap \bar{U} = \{x_*\}. \tag{1.203}$$

Lemma 1.1. *If* $x_* \in \partial U$ *has the exterior sphere property, then* x_* *is regular.*

Proof. Let $u(x) = \Phi(x_* - y) - \Phi(x - y)$. Then by definition $u(x) = 0$ if $x = x_*$, and $u(x) > 0$ if $x \in U$. Moreover, u is super-harmonic for $y \in U$ (in fact it is harmonic). Therefore we explicitly constructed a barrier at x_*, which implies that x_* is regular. □

Remark 1.18. *Any set* U *with boundary* ∂U *of class* C^2 *has the exterior sphere property.*

Lemma 1.2. *Let* $U \subset \mathbb{R}^n$ *open and bounded,* $g \in C^0(\partial U)$. *Let*

$$u(x) = \sup\{v(x) \mid v \in S_g\}. \tag{1.204}$$

If $x_* \in \partial U$ *is regular, then*

$$\lim_{x \to x_*} u(x) = u(x_*) = g(x_*) \tag{1.205}$$

Proof. Let $\epsilon > 0$. There exists $\delta > 0$ such that

$$|g(x) - g(x_*)| < \epsilon, \qquad \forall x \in B(x_*, \delta) \cap \partial U. \tag{1.206}$$

Therefore, $\forall x \in \partial U$:

$$|g(x) - g(x_*)| \leqslant \epsilon + 2 \underbrace{\frac{\max_{\partial U} |g|}{\min_{\bar{U}\backslash B(x_*,\delta)} w|}}_{=:C} w(x) \tag{1.207}$$

where w appears without the absolute value since $w \geqslant 0$. We then have:

$$g(x_*) - Cw(x) - \epsilon \leqslant g(x) \leqslant g(x_*) + Cw(x) + \epsilon, \qquad \forall x \in \partial U. \tag{1.208}$$

The function $g(x_*) - Cw(x) - \epsilon$ is subharmonic and belongs to S_g. Therefore, $\forall x \in \bar{U}$, $g(x_*) - Cw(x) - \epsilon \leqslant u(x)$. Also, the function $g(x_*) + Cw(x) + x$ is super-harmonic. This implies that $\forall v \in S_g$

$$v(x) \leqslant g(x) \leqslant g(x_*) + Cw(x) + \epsilon \qquad \forall x \in \partial U, \tag{1.209}$$

and by the max principle

$$v(x) \leqslant g(x_*) + Cw(x) + \epsilon \qquad \forall x \in U. \tag{1.210}$$

Therefore

$$\begin{aligned} u(x) = \sup\{v(x) \big| v \in S_g\} &\leqslant g(x_*) + Cw(x) + \epsilon, \qquad \forall x \in U \Rightarrow \\ \Rightarrow \lim_{x \to x_*} \sup |u(x) - g(x_*)| &\leqslant \epsilon + \lim_{x \to x_*} Cw(x) = \epsilon, \qquad \forall \epsilon > 0. \end{aligned} \tag{1.211}$$

□

Theorem 1.16. *Let $U \subset \mathbb{R}^n$ open and bounded. The boundary value problem*

$$\begin{cases} \Delta u = 0 & \text{in } U, \\ u = g & \text{on } \partial U \end{cases} \tag{1.212}$$

with $g \in C^0(\partial U)$ admits a solution if and only if all points $x_* \in \partial U$ are regular.

Proof. Let all $x_* \in \partial U$ be regular. Then, existence follows from Theorem (1.15) and Lemma (1.2). Suppose now that the Dirichlet boundary problem admits a solution $\forall g \in C^0(\partial U)$, and let $x_* \in \partial U$. Then, the solution of $\Delta U = 0$, $u(x) = |x - x_*|$ on ∂U is a barrier at x_*. By the maximum principle, $u \geqslant 0$ on U. Suppose now that $y \in U$ such that $u(y) = 0$. Then, $u(y) = 0$ in U since $0 = \inf_{x \in \partial U} u(x)$ by the strong maximum principle. But this is a contradiction, since

$$u(x) \neq \text{ constant on } \partial U,$$

and $u \in C^2(U) \cap C(\bar{U})$ which implies that $u(x) > 0$ for all $x \in \bar{U}, x \neq x_*$. □

To conclude, we would like to extend the previous results to prove existence of solutions for the nonhomogeneous case (1.100). Recall the definition of Green's function, $G(x, y) = \Phi(x - y) - \phi^x(y)$ with

$$\begin{cases} \Delta \phi^x = 0 & \text{in } U, \\ \phi^x(y) = \Phi(x - y) & \text{on } \partial U \end{cases} \tag{1.213}$$

We look for a solution u of (1.100) as $u = u_1 + u_2$, where u_1 and u_2 are defined as follows. We have:

$$\begin{cases} \Delta u_2 = 0 & \text{in } U, \\ u_2 = g & \text{on } \partial U, \end{cases} \tag{1.214}$$

which admits a unique solution, as we proved, while

$$\begin{cases} \Delta u_1 = f & \text{in } U, \\ u_1 = 0 & \text{on } \partial U, \end{cases} \tag{1.215}$$

which also admits a unique solution, given by:

$$u_1(x) = \int_U f(y) G(x - y) dy. \tag{1.216}$$

1.3.9 Energy methods

Let us consider:

$$\begin{cases} -\Delta u = f & \text{in } U, \\ u = g & \text{on } \partial U, \end{cases} \tag{1.217}$$

with $U \subset \mathbb{R}^n$ open, bounded and such that ∂U is of class C^1. Here we shall give a different proof of the following theorem.

Theorem 1.17. *The boundary value problem admits at most one solution $u \in C^2(\bar{U})$.*

Proof. Suppose u_1 and u_2 are two solutions. Then, $w = u_1 - u_2$ is a solution of:

$$\begin{cases} \Delta w = 0 & \text{in } U, \\ w = 0 & \text{in } \partial U, \end{cases} \tag{1.218}$$

and this implies that

$$0 = -\int_U w\Delta w dx = \int_U |\nabla w|^2 dx, \tag{1.219}$$

which is true if and only if $|\nabla w| = 0$ in U, meaning that w is constant in U. Being $w = 0$ on ∂U, then $w = 0$ in U. □

Remark 1.19. *The functional*

$$I(w) = \int_U \frac{1}{2}|\nabla w|^2, \tag{1.220}$$

is called the energy of the solution. In the more general case of the Poisson initial value problem (1.100)*, the energy functional would read as*

$$I(w) := \int_U dx \frac{1}{2}|\nabla w|^2 - fw \tag{1.221}$$

defined on the domain

$$\mathcal{A} := \{w \in C^2(U) \big| w = g \text{ on } \partial U\}. \tag{1.222}$$

Theorem 1.18 (Dirichlet's principle)**.** *Suppose that $u \in C^2(\bar{U})$ solves the initial value problem* (1.186)*. Then*

$$I(u) = \min_{w \in \mathcal{A}} I(w). \tag{1.223}$$

Conversely, if $u \in \mathcal{A}$ satisfies (1.223)*, then u solves the initial value problem* (1.186)*.*

Proof. Suppose that u is a solution of the initial value problem. Let $w \in \mathcal{A}$. Then

$$0 = \int_U (-\Delta u - f)(u - w)dx. \tag{1.224}$$

Integrating by parts we get:

$$0 = \int_U (Du \cdot D(u - w) - f(u - w))dx, \tag{1.225}$$

where the boundary terms vanish since $u - w \restriction_{\partial U} = g - g = 0$. Hence,

$$\int_u |Du|^2 - uf = \int Du \cdot Dw - wf \leqslant \frac{1}{2}\int_U |Du|^2 + \frac{1}{2}\int |Dw|^2 - wf = \int Du \cdot Dw - wf. \tag{1.226}$$

Thus, we have

$$\frac{1}{2}\int_U |Du|^2 - uf \leqslant \frac{1}{2}\int |Dw|^2 - wf. \tag{1.227}$$

Therefore, since $u \in \mathcal{A}$, u is the minimizer of I on $\mathcal{A}$. Now we want to prove that if u is the minimizer of I, then it solves the boundary value problem. Let $v \in C_c^\infty(U)$, and define

$$i(\tau) := I(u + 2v), \qquad \tau \in \mathbb{R}. \tag{1.228}$$

Notice that $u+2v \in \mathcal{A}$ for all τ. Being u by assumption the minimizer of I, $\tau \to i(\tau)$ has a minimum at $\tau = 0$. Suppose now that $i(\tau)$ is differentiable at $\tau = 0$, then

$$\frac{d}{d\tau} i(\tau)\big|_{\tau=0} = 0. \tag{1.229}$$

Let us now compute the derivative and check that it exists:

$$i(\tau) = \int_U \frac{1}{2}|Du + 2Dv|^2 - (u+2v)f dx = \int_U \frac{1}{2}|Du|^2 + 2Du \cdot Dv + \frac{2^2}{2}|Dv|^2 - (u+2x)f, \tag{1.230}$$

so that

$$\frac{d}{d\tau} I(\tau) = \int_U Du \cdot Dv + 2|Dv|^2 - vf. \tag{1.231}$$

Being $u, v \in C^2(\bar{U})$ the derivative exists. Moreover, if we impose the condition (1.229), we find

$$0 = \frac{d}{d\tau} i(\tau)\big|_{\tau=0} = \int_U Du \cdot Dv - vf dx = \int_U (-\Delta u - f) v dx \tag{1.232}$$

where we used that $v = 0$ on ∂U. Being the identity valid $\forall v \in C_c^\infty(U)$, the latter equality can be true only if $-\Delta u = f$ in U. □

Remark 1.20. *This theorem tells us that the uniqueness of the solution of the initial value problem is equivalent to the uniqueness of the minimizer of $I(\cdot)$ on $\mathcal{A}$.*

1.4 Heat equation

In this section, we will be interested in understanding the so called heat equation, in both the homogeneous and in the non homogeneous case:

$$u_t - \Delta u = 0, \tag{1.233}$$
$$u_t - \Delta u = f, \tag{1.234}$$

where $t > 0$, $x \in U \subset \mathbb{R}^n$ open. The unknown is $u(x,t)$ and the Laplace operator acts only on the variables $(x_1, \cdots, x_n)$.

Motivations. The heat equation describes a time dependent phenomenon: one can think as $x \in \mathbb{R}^n$ to be a space variable and t a time variable. The function $f(x,t)$ is also given. Let $F \equiv F(x,t)$ be a *local flux* through x, and

$$\frac{d}{dt} \int_V u dx = -\int_{\partial V} F \cdot \nu DS \Rightarrow u_t = -\mathrm{div} F \tag{1.235}$$

since V is arbitrary. Phenomenologically, the flux is directed from regions with high concentration to regions with low concentration, which motivates the choice:

$$F = -aDu, \qquad a > 0\ . \tag{1.236}$$

Thus,

$$u_t = a\mathrm{div}(Du) = a\Delta u. \tag{1.237}$$

1.4.1 Fundamental solution

For the Laplace equation, we found the fundamental solution by looking at radial solutions. Here, we notice that if u solves $u_t - \Delta u = 0$, then

$$u_t^\lambda := u^\lambda(x,t) = u(\lambda x, \lambda^2 t), \tag{1.238}$$

solves

$$u_t^\lambda - \Delta u^\lambda = 0. \tag{1.239}$$

Therefore, one would like to find a solution such that

$$u^\lambda = u, \qquad \forall \lambda \tag{1.240}$$

It is convenient to be a bit more general, and to look for solutions invariant under the rescaling

$$u(x,t) = \lambda^\alpha u(\lambda^\beta x, \lambda t), \qquad \forall \lambda > 0, \tag{1.241}$$

for some α and β. Let $\lambda = t^{-1}$. The latter condition reads:

$$u(x,t) = t^{-\alpha} u(t^{-\beta} x, 1), \tag{1.242}$$

and the homogeneous heat equation (1.233) becomes:

$$\begin{aligned}
&-\alpha t^{-\alpha-1} u(t^{-\beta}x,1) + t^{-\alpha-1} u(t^{-\beta}x,1) + \\
&+t^{-\alpha}(-\beta) t^{-\beta-1} \left(x \cdot D_x u\right) \left(t^{-\beta}x,1\right) - t^{-\alpha} t^{-2\beta} (Du) \left(t^{-\beta}x,1\right) = 0 \Leftrightarrow \\
\Leftrightarrow &\alpha t^{-(\alpha+1)} u(y,1) + t^{-(\alpha+1)} \beta (y D_y u)(y,1) + \\
&+t^{\alpha+2\beta} (\Delta u)(y,1) = 0
\end{aligned} \tag{1.243}$$

Let now $\beta = \frac{1}{2}$. The coefficient $t^{-(\alpha+1)}$ factors out and, calling $u(\cdot,1) \equiv v(\cdot)$, we are left with

$$\alpha v(y) + \frac{1}{2} y \cdot D_y v + \Delta v = 0. \tag{1.244}$$

Let us further assume that v is radial

$$v(x) \equiv v(|x|). \tag{1.245}$$

Proceeding as for the Laplace equation, we get

$$\alpha v(y) + \frac{1}{2} y \cdot \frac{y}{|y|} v'(|y|) + \underbrace{v''(|y|) + \frac{n-1}{r} v'(y)}_{\text{Laplacian of } v} = 0. \tag{1.246}$$

Setting $w(r) = v(x)$, $r = |x|$ the latter equation reads:

$$\alpha w + \frac{1}{2} r w' + w'' + \frac{n-1}{r} w' = 0. \tag{1.247}$$

Then, fixing $\alpha = n/2$

$$\frac{n}{2} w + \frac{r}{2} w' + w'' + \frac{n-1}{r} w' = 0. \tag{1.248}$$

Now, notice that

$$r^{n-1}\left(w'' + \frac{n-1}{r}w'\right) = \left(r^{n-1}w'\right)', \tag{1.249}$$

$$r^{n-1}\left(\frac{n}{2}w + \frac{r}{2}w'\right) = \frac{1}{2}\left(r^n w\right)' \tag{1.250}$$

Hence, (1.248) implies

$$(r^{n-1}w')' + \frac{1}{2}(r^n w)' = 0 \Rightarrow r^{n-1}w' + \frac{1}{2}r^n w = a, \qquad \text{for } a \text{ constant.} \tag{1.251}$$

Suppose now that $w, w' \underset{r\to\infty}{\longrightarrow} 0$ fast enough. Then $a = 0$, hence

$$w' = -\frac{1}{2}rw. \tag{1.252}$$

Therefore, we reduced the problem to an ordinary differential equation, whose solution is

$$w = be^{-\frac{r^2}{4}}, \tag{1.253}$$

for some constant b. Summarizing

$$u(x,t) = t^{-\alpha}u(t^{-\beta}x, 1) \equiv t^{-\alpha}w(t^{-\beta}|x|) = t^{-\frac{n}{2}}be^{-\frac{|x|^2}{4t}}. \tag{1.254}$$

Definition 18 (Fundamental solution of the heat equation)**.** *The function*

$$\begin{cases} \Phi(x,t) = \frac{1}{(4\pi t)^{n/2}}e^{-\frac{|x|^2}{4t}}, & x \in \mathbb{R}^n, t > 0, \\ 0 & x \in \mathbb{R}^n, t < 0, \end{cases} \tag{1.255}$$

is the fundamental solution of the heat equation.

Remark 1.21. *Notice that, for $x \neq 0$,*

$$\lim_{t\to 0^+} \Phi(x,t) = 0. \tag{1.256}$$

Instead, Φ is singular at $(0,0)$. The normalization constant has been chosen in such a way that

$$\int_{\mathbb{R}^n} \Phi(x,t)dx = 1. \tag{1.257}$$

Consider now the initial value problem

$$\begin{cases} u_t - \Delta u = 0, & \text{in } \mathbb{R}^n \times (0,\infty), \\ u = g, & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{1.258}$$

For $t > 0$, the function $(x,t) \to \Phi(x-y,t)$ is perfectly regular. Therefore, analogously to the Laplace initial value problem case, one expects

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)g(y)dy, \tag{1.259}$$

to be a solution of the heat equation. In fact, this is true for $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

Theorem 1.19 (Solution of the initial value problem)**.** *Let $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Let*

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)g(y)dy \equiv \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} dy e^{-\frac{|x-y|^2}{4t}} g(y), \qquad \forall x \in \mathbb{R}^n, t > 0. \tag{1.260}$$

Then,

1. $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$,
2. $u_t - \Delta u = 0, \qquad x \in \mathbb{R}^n, t > 0,$
3. $\lim_{\substack{(x,t)\to(x_0,0) \\ x\in\mathbb{R}^n, t>0}} u(x,t) = g(x_0), \qquad \forall x_0 \in \mathbb{R}^n.$

Remark 1.22. *This shows that u in (1.260) is a solution of the initial value problem (1.258).*

Proof. 1. It follows from the fact that $\Phi(x-y,t)$ is C^∞ in $x \in \mathbb{R}^n$, $t > 0$ and the derivatives are absolutely integrable.

2. It follows from the fact that Φ is a solution.

3. Let $x_0 \in \mathbb{R}^n$, and fix $\epsilon > 0$. Being g continuous, there exists $\delta > 0$ such that

$$|g(y) - g(x_0)| \leqslant \epsilon \text{ if } |y - x_0| < \delta, y \in \mathbb{R}^n. \tag{1.261}$$

Take now $|x - x_0| < \delta/2$: so we have

$$|u(x,t) - g(x_0)| \leqslant \int_{\mathbb{R}^n} \Phi(x-y,t)|g(x) - g(x_0)|, \tag{1.262}$$

where we used that $\int dy\, \Phi(x-y,t) = 1$. Then,

$$\begin{aligned} |u(x,t) - g(x_0)| \leqslant & \int_{\mathbb{R}^n \setminus B(x_0,\delta)} \Phi(x-y,t)|g(y) - g(x_0)|dy + \\ & + \int_{B(x_0,\delta)} \Phi(x-y)|g(y) - g(x_0)|dy \equiv I + II \end{aligned} \tag{1.263}$$

By continuity, $II \leqslant \epsilon$. Thus we are left with studying the term I, corresponding to $|y - x_0| \geqslant \delta$. We use:

$$|y - x_0| = |y - x + x - x_0| \leqslant |y - x| + \delta/2 \leqslant |y - x| + \frac{1}{2}|y - x_0|, \tag{1.264}$$

meaning that $|y - x| \geqslant \frac{1}{2}|y - x_0|$. Therefore

$$\begin{aligned} I &\leqslant 2\|g\|_\infty \int_{\mathbb{R}^n \setminus B(x_0,\delta)} \Phi(x-y,t)dy \leqslant \frac{c}{t^{n/2}} \int_{\mathbb{R}^n \setminus B(x_0,\delta)} e^{-\frac{|x-y|^2}{4t}} dy \leqslant \\ &\leqslant \frac{c}{t^{n/2}} \int_{\mathbb{R}^n \setminus B(x_0,\delta)} e^{-\frac{|x_0-y|^2}{16t}} dy = c \int_{\mathbb{R}^n \setminus B(0,\delta/\sqrt{t})} e^{-\frac{|z|^2}{16}} dz \underset{t\to 0^+}{\longrightarrow} 0, \end{aligned} \tag{1.265}$$

where in the latter step we performed the change on variables $(y - x_0)/\sqrt{t} = z$. So, for $|x - x_0| < \delta/2$ and $t > 0$ small enough,

$$|u(x,t) - g(x_0)| \leqslant 2\epsilon. \tag{1.266}$$

□

Remark 1.23. *The solution has infinite propagation speed: as soon as $t > 0$, $u(x,t)$ is everywhere, even if g is compactly supported.*

We are now ready to discuss the solution of the non-homogeneous problem:

$$\begin{cases} u_t - \Delta u = f, & \text{in } \mathbb{R}^n \times (0,\infty), \\ u = 0 & \text{on } \mathbb{R}^n \times \{0\}. \end{cases} \tag{1.267}$$

We would like to find a solution of the problem starting from the solution of the homogeneous equation. Before writing the expression and checking it, let us first discuss some heuristics behind it: the solution of the non homogeneous equation is found via the Duhamel principle, which is a very useful trick in PDEs.
Suppose that $u(x,t)$ is a solution of the problem and, at a given time $s \leqslant t$, evolve $u(\cdot, s)$ with the homogeneous equation ($f = 0$), and let us call $\tilde{u}(x, s'; s)$ the solution of

$$\begin{cases} \tilde{u}_{s'} - \Delta \tilde{u} = 0, & \text{in } \mathbb{R}^n \times \{s > s'\}, \\ \tilde{u} = u, & \text{on } \mathbb{R}^n \times \{s' = s\}. \end{cases} \tag{1.268}$$

Consider $\tilde{u}(x,t;s)$: in general, $\tilde{u}(x,t;s) \neq u(x,t)$. They would be the same if there was no non-homogeneity. Therefore, one expects $\frac{d}{dt}\tilde{u}(x,t;s)$ to depend on f. We compute

$$\frac{d}{ds}\tilde{u}(x,t;s) = \frac{d}{ds}\int_{\mathbb{R}^n} dy \Phi(x-y, t-s) u(y,s). \tag{1.269}$$

$u(\cdot, s) \equiv g$ is the initial condition. By what we proved before

$$\lim_{s \to t^-} \tilde{u}(x,t;s) = u(y,t). \tag{1.270}$$

Also

$$\lim_{s \to 0} \tilde{u}(x,t;s) = 0, \tag{1.271}$$

since $u(y,0) = 0$ by assumption. Therefore,

$$\begin{aligned} u(y,t) &= \int_0^t ds \frac{d}{ds}\tilde{u}(x,t;s) = \int_0^t ds \frac{d}{ds}\int dy \Phi(x-y,t-s)u(y,s) = \\ &= \int_0^t ds \int dy \left[\left(\frac{d}{ds}\Phi(x-y,t-s) \right) u(y,s) + \Phi(x-y,t-s)\frac{d}{ds}u(y,s) \right] = \\ &= \int_0^t ds \int dy \left[-\frac{d}{dt}\Phi(x-y,t-s)u(y,s) + \Phi(x-y,t-s)\frac{d}{ds}u(y,s) \right] = \\ &= \int_0^t ds \int dy \left[-\Delta_y \Phi(x-y,t-s)u(y,s) + \Phi(x-y,t-s)\left(\Delta_y u(y,s) + f(y,s)\right) \right] = \\ &= \int_0^t ds \int dy \left[\Phi(x-y,t-s)\left(-\Delta_y u(y,s) + \Delta_y u(y,s) + f(y,s)\right) \right] = \\ &= \int_0^t ds \int dy \Phi(x-y,t-s) f(y,s) = u(x,t). \end{aligned} \tag{1.272}$$

In this way we have a guess for the solution of the non homogeneous initial value problem (1.267). Of course, we have to prove that (1.272) is indeed a solution of the partial differential equation.

Theorem 1.20. *Let $f \in C_1^2(\mathbb{R}^n \times [0\infty)) = \{f : \mathbb{R}^n \to \mathbb{R} \big| f$ is $C^2(\mathbb{R}^2)$ in x, and $C^1(0,\infty)$ in $t\}$ and suppose that f has compact support on (x,t). Let u be given by (1.272). Then*

1. $u \in C_1^2(\mathbb{R}^n \times (0,\infty))$.
2. $u_t(x,t) - \Delta u(x,t) = f(x,t)$, $x \in \mathbb{R}^n, t > 0$.
3. $\lim_{(x,t)\to(x_0,0) x\in\mathbb{R}^n, t>0} u(x,t) = 0$, $\forall x_0 \in \mathbb{R}^n$.

Proof. 1. Φ has a singularity at $(0,0)$: we cannot exchange directly the integral and the derivatives ∂_t, Δ. However, we can change variable in the integral, so that the derivatives only act on f:

$$u(x,t) = \int_0^t ds \int_{\mathbb{R}^n} dy \Phi(x-y, t-s) f(y,s) = \int_0^t ds \int_{\mathbb{R}^n} dy \Phi(y,s) f(x-s, t-s). \tag{1.273}$$

Recall that $f \in C_1^2(\mathbb{R}^n \times [0,\infty))$. We compute

$$u_t = \int_{\mathbb{R}^n} dy \Phi(y,t) f(x-y,0) + \int_0^t ds \int_{\mathbb{R}^n} dy \phi(y,s) f_t(x-y,t-s), \tag{1.274}$$

and

$$u_{x_i x_j} = \int_0^t ds \int_{\mathbb{R}^n} dy \Phi(y,s) f_{x_i x_j}(x-y, t-s), \tag{1.275}$$

which exists since f has compact support.

2.

$$\begin{aligned} u_t - \Delta u &= \int_0^t ds \int_{\mathbb{R}^n} dy \Phi(y,s) \left(\partial_t - \Delta_x\right) f(x-y,t-s) + \int_{\mathbb{R}^n dy \Phi(y,t) f(x-y,0)} = \\ &= \int_\epsilon^t ds \int dy_{\mathbb{R}^n} \Phi(y,s) \left(-\partial_s - \Delta_y\right) f(x-y,t-s) + \\ &+ \int_0^\epsilon ds \int_{\mathbb{R}^n} dy \Phi(y,s) \left(-\partial_s - \Delta_y\right) f(x-y,t-s) + \\ &+ \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0) dy \equiv I + II + III \end{aligned} \tag{1.276}$$

Let us consider the three terms separately:

$$|II| \leqslant \left(\|f_t\|_\infty + \|D^2 f\|_\infty\right) \int_0^\epsilon ds \underbrace{\left(\int dy \Phi(y,s)\right)}_{=1} \leqslant C\epsilon. \tag{1.277}$$

Now we will see that a piece of the I compensates III:

$$\begin{aligned} I &= \int_\epsilon^t \int_{\mathbb{R}^n} dy \left(\left(\partial_s - \Delta_y\right) \Phi(y,s)\right) f(x-y,t-s) + \\ &+ \int_{\mathbb{R}^n} \Phi(y,\epsilon) f(x-y,t-\epsilon) dy - \\ &- \int_{\mathbb{R}^n} \Phi(y,\epsilon) f(x-y,0) dy \equiv \\ &\equiv \int_{\mathbb{R}^n} \Phi(y,\epsilon) f(x-y,t-\epsilon) dy - III \end{aligned} \tag{1.278}$$

being $(\partial_s - \Delta)\Phi = 0$. All in all,

$$u_t(x,t) - \Delta u(x,t) = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^n} \Phi(y,\epsilon) f(x-y, t-\epsilon) dy = f(x,t), \qquad x \in \mathbb{R}^n, t > 0. \tag{1.279}$$

3. We are left with checking the boundary condition $u(x,0) = 0$. We have:

$$\begin{aligned} \|u(\cdot,t)\|_\infty \leqslant & \int_0^t ds \int_{\mathbb{R}^n} |\Phi(x-y, t-s)||f(y,s)|dy \leqslant \\ \leqslant & \|f\|_\infty \int_0^t ds \int_{\mathbb{R}^n} dy \Phi(y-x, t-s) \leqslant t\|f\|_\infty \underset{t \to 0^+}{\longrightarrow} 0. \end{aligned} \tag{1.280}$$

□

Remark 1.24. *By the linearity of the heat equation, we have that*

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y) g(y) dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s) dy ds \tag{1.281}$$

is a solution of

$$\begin{cases} u_t - \Delta u = f, & \text{in } \mathbb{R}^n \times (0,\infty), \\ u = g, & \text{on } \mathbb{R}^n \times \{0\}. \end{cases} \tag{1.282}$$

1.4.2 Mean-value formula

Definition 19 (Parabolic cylinder and parabolic boundary)**.** *Let $U \subset \mathbb{R}^n$ open and bounded. Let $T > 0$ be a fixed time. A parabolic cylinder is the set*

$$U_T := U \times (0, T]. \tag{1.283}$$

Also, we define the parabolic boundary of U_T as

$$\Gamma_T := \bar{U}_T \backslash U_T. \tag{1.284}$$

Remark 1.25. *To formulate the analog of the mean value theorem for the harmonic functions (1.2), we need to introduce the analog of the ball $B(x,r)$ for the Laplace equation. Notice that $\partial B(x,r)$ is a level set for the fundamental solution of the Laplace equation $\Phi(x-y)$, $\Phi(x-y) = r$ on $\partial B(x,r)$.*

Definition 20 (Heat ball)**.** *For $(x,t) \in \mathbb{R}^{n+1}$, $t > 0$ and for $r > 0$, we define*

$$E(x,t,r) := \left\{ (y,s) \in \mathbb{R}^{n+1} \middle| \Phi(x-y, t-s) > \frac{1}{r^n} \right\} \tag{1.285}$$

Remark 1.26. *The set $E(x,t,s)$ does not contain points (y,s) with $s > t$, since for these points $\Phi(x-y, t-s) = 0$. Also, the point (x,t), where Φ is not defined, is on ∂E. Moreover, since*

$$\Phi(x-y, t-s) \underset{\substack{s \to -\infty \\ |x-y| \to \infty}}{\longrightarrow} 0,$$

the set E is bounded. ∂E is regular away from (x,t).

Theorem 1.21 (Mean value formula). *Let $u \in C^2(U_T)$ be a solution of the heat equation (1.258). Then,*

$$u(x,t) = \frac{1}{4r^n} \int_{E(x,t;r)} dyds\, u(y,s) \frac{|x-y|^2}{|t-s|^2}, \tag{1.286}$$

$\forall E(x,t;r) \subset U_T$.

Proof. Without loss of generality, let $x = 0$ and $t = 0$. Let $E(r) = E(0,0;r)$ and

$$\Phi(r) = \frac{1}{r^n} \int_{E(r)} u(y,s) \frac{|y|^2}{s^2} dyds = \int_{E(1)} u(ry, r^2 s) \frac{|y|^2}{s^2} dyds. \tag{1.287}$$

As in the proof of the mean value theorem (1.2) for Laplace's equation, we will show that $\Phi(r)$ is constant in r. We compute:

$$\begin{aligned} \frac{d}{dr}\Phi(r) &= \frac{1}{E(1)} \left[y \cdot Du \frac{|y^2|}{s^2} + 2ru_s \frac{|y|^2}{s} \right] dyds = \\ &= \frac{1}{r^{n+1}} \int_{E(r)} \left[y \cdot Du(y,s) \frac{|y|^2}{s^2} + 2u_s \frac{|y|^2}{s} \right] \equiv A + B \end{aligned} \tag{1.288}$$

Let us define

$$\psi := -\frac{n}{2} \log(-4\pi s) + \frac{|y|^2}{4s} + n \log r, \tag{1.289}$$

which in particular vanishes on $\partial E \backslash \{(0,0)\}$ since $\Phi(y,-s) = \frac{1}{r^n}$. We use ψ to write

$$\begin{aligned} B =& \frac{1}{r^{n+1}} \int_{E(r)} dyds 2u_s \frac{|y|^2}{s} = \\ =& \frac{1}{r^{n+1}} \int_{E(r)} dyds 4u_s \sum_{i=1}^n y_i \psi_{y_i} = -\frac{1}{r^{n+1}} \int_{E(r)} dyds \left[4nu_s\psi + 4\sum_{i=1}^n y_i \cdot u_s y_i \psi \right] = \\ =& -\frac{1}{r^{n+1}} \int_{E(r)} dyds \left[-4nu_s\psi + 4\sum_{i=1}^n y_i \cdot u_{y_i} \psi_s \right] = \\ =& \frac{1}{r^{n+1}} \int_{E(r)} dyds \left[-4nu_s\psi - \frac{2n}{s} \sum_{i=1}^n u_{y_i} y_i \right] - A, \end{aligned} \tag{1.290}$$

where we used Gauss-Green theorem and the integration by parts exploiting the fact that $\psi = 0$ on ∂E. Therefore

$$\begin{aligned} \phi'(r) =& A + B = \frac{1}{r^{n+1}} \int_{E(r)} dyds \left[-4nu_s\psi - \frac{2n}{s} Du \cdot y \right] = \\ =& \frac{1}{r^{n+1}} \int_{E(r)} dyds \left[-4n\Delta u\psi - \frac{2n}{s} Du \cdot y \right] = \\ =& \frac{1}{r^{n+1}} \int_{E(r)} dyds \left[4nDu \cdot D\psi - \frac{2n}{s} Du \cdot y \right] = 0, \end{aligned} \tag{1.291}$$

being, in the last line, $D\psi = \frac{y}{2s}$. Therefore, ϕ is constant and

$$\phi(r) = \lim_{a\to 0^+} \phi(a) = \lim_{a\to 0^+} \frac{1}{a^n} \int_{E(a)} u(y,s) \frac{|y|^2}{s^2} dyds. \tag{1.292}$$

The set $\overline{E(a)}$ shrinks to $(0,0)$ as $r \to 0$, and ϕ is always bounded for $(y,s) \neq (0,0)$. Therefore,

$$(1.292) = u(0,0) \lim_{a\to 0^+} \frac{1}{a^n} \int_{E(a)} \frac{|y|^2}{s^2} dyds = \int_{E(1)} \frac{|y^2|}{s^2} dyds \tag{1.293}$$

since u is continuous. Finally, since

$$\int_{E(1)} dyds \frac{|y|^2}{s^2} = 4, \tag{1.294}$$

the claim is proved. ☐

1.4.3 Maximum principle

We can use the mean value formula to prove the strong maximum principle for the heat equation, which in turn can be used to prove the uniqueness of the solution of the heat equation.

Theorem 1.22. *Let $U \subset \mathbb{R}^n$ open and bounded, and suppose $u \in C_1^2(U_T) \cap C(\bar{U}_T)$ solves the heat equation in U_T. Then:*

1. *Maximum principle:*
$$\max_{\bar{U}_T} u = \max_{\Gamma_T} u. \tag{1.295}$$

2. *Strong maximum principle: let U be also connected. Suppose that there exists a point $(x_0, t_0) \in U_T$ such that $u(x_0, t_0) = \max_{\bar{U}_T} u$. Then, u is constant in U_{t_0}.*

Remark 1.27. *The solution might however change after t_0, provided the boundary condition $u = g$ on Γ_T changes after t_0. This is natural: what happens at t_0 is only influenced by what happens before t_0.*

Proof. Suppose that there exists a point $(x_0, t_0) \in U_T$ with $u(x_0, t_0) = M = \max_{\bar{U}_T} u$. Then, for r small enough, $E(x_0, t_0; r) \subset U_T$. By the mean value formula

$$M = u(x_0, t_0) = \frac{1}{4r^n} \int_{E(x_0,t_0;r)} u(y,s) \frac{|x_0 - y|^2}{|t_0 - s|^2} dyds \leqslant M \frac{1}{4r^n} \int_{E(x_0,t_0;r)} \frac{|x_0 - y|^2}{|t_0 - s|^2} dyds = M. \tag{1.296}$$

The equality holds if $u = M$ in $E(x_0, t_0; r)$. Now, we are left with extending the result to the whole U_T: we will show that the function is equal to M in z, $\forall z \in U_{t_0}$ by proving that it is equal to M on all the paths connecting (x_0, t_0) with (z_0, s_0). It is convenient to look at paths made of the union of connected segments. To begin, let us show that U is constant for any segment L connecting (x_0, t_0) to any other point (y_0, s_0) in U_T. Let

$$r_0 := \min\{s \geqslant s_0 \big| u(x,t) = M \quad \forall (x,t) \in L, s \geqslant t \geqslant t_0\}. \tag{1.297}$$

We claim that $r_0 = s_0$, *i.e.* that u is constant over the whole line L. Suppose it is false. Then, $u(z_0, r_0) = M$ for some $(z_0, r_0) \subset L \cap U_T$, $(r_0 > s_0)$, $\Rightarrow u \equiv M$ in $E(z_0, r_0; r)$, by the mean value formula and by the previous argument. But $E(z_0, r_0) \supset L \cap \{r_0 - \sigma \leqslant t \leqslant r_0\}$, for some $\sigma > 0$, that is a contradiction: so $r_0 = s_0$. To prove that u is constant for all points in U_T, we connect (x_0, t_0) to any point $(z_0, s_0) \in U_{t_0}$ by choosing an appropriate polygonal path. This proves the strong maximum principle. The max principle follows from the strong maximum principle for all the connected components of U_T. □

Remark 1.28. *Consider the heat equation in a bounded domain U_T, with U open, bounded and connected. Let $u \in C_1^2(U_T) \cap C(\bar{U}_T)$ be a solution of*

$$\begin{cases} u_t - \Delta u = 0 & \text{in } U_T, \\ u = 0 & \text{on } \partial U \times [0, T], \\ u = g & \text{on } U \times \{t = 0\}. \end{cases} \tag{1.298}$$

Suppose that $g \geqslant 0$: then $u > 0$ everywhere if g is positive somewhere on U (this means that there is an infinite propagation speed!). If $g > 0$ on U, there is nothing to prove thanks to the maximum (minimum) principle. Suppose that there exists $x_* \in U$ such that $g(x_*) = 0$. Suppose that there exists $(z_*, t_*) \in U_T$ such that $u(z_*, t_*) = 0$. Then, by the minimum principle, $u(x, t) = 0$ for any $(x, t) \in U_T$, which is impossible since $g(y_*) > 0$ for some $y_* \in U$ and u is continuous.

The maximum principle can be immediately used to prove that the solution of the heat equation in bounded domains is unique.

Theorem 1.23 (Uniqueness of the solution of the heat equation initial value problem)**.** *Let $g \in C(\Gamma_T)$, $f \in C(U_T)$. Then, there exist at most one solution $u \in C_1^2(U_T) \cap C(\bar{U}_T)$ of*

$$\begin{cases} u_t - \Delta u = f & \text{in } U_T, \\ u = g & \text{on } \Gamma_T. \end{cases} \tag{1.299}$$

Proof. Let $u, \tilde{u}$ be two solutions of the heat equation initial value problem. Let $w = \pm(u - \tilde{u})$. We have:

$$\begin{cases} w_t - \Delta w = 0 & \text{in } U_T, \\ w = 0 & \text{on } \Gamma_T. \end{cases} \tag{1.300}$$

By the maximum principle,

$$\max_{\bar{U}_T} w = \max_{\Gamma_T} w = 0 \Rightarrow u = \tilde{u}. \tag{1.301}$$

□

Let us now drop the boundedness condition for the domain, and consider the heat equation on $\mathbb{R}^n$. It turns out that, if one focuses to a suitable class of solutions, uniqueness holds true. In the case of Laplace's equation, a crucial ingredient if the proof of uniqueness among bounded functions was the Liouville theorem. Here, we shall instead rely on a version of the maximum principle for unbounded domains.

Theorem 1.24 (Maximum principle for the Cauchy problem). *Suppose that $u \in C^2_1(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ is a solution of*

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\}, \end{cases} \tag{1.302}$$

and satisfies $u(x,t) \leqslant Ae^{a|x|^2}$ for $x \in \mathbb{R}^n$, $0 \leqslant t \leqslant T$ for some $A, a > 0$. Then

$$\sup_{\mathbb{R}^n \times [0,T]} u = \sup_{\mathbb{R}^n} g. \tag{1.303}$$

Proof. At first, suppose that $4aT < 1$. This in particular means that $4a(T+\epsilon) < 1$ for ϵ small enough. Let $y \in \mathbb{R}^n$, $\mu > 0$, define

$$v(x,t) := u(x,t) - \frac{\mu}{(T+\epsilon-t)^{\frac{n}{2}}} e^{\frac{|x-y|^2}{4(T+\epsilon-t)}}. \tag{1.304}$$

Then, one can check that $v_t - \Delta v = 0$ in $\mathbb{R}^n \times (0,T]$. Let now $r > 0$, $U := B(y,r)$, $U_T := B(y,r) \times (0,T]$: by the maximum principle we get

$$\max_{\bar{U}_T} v = \max_{\Gamma_T} v. \tag{1.305}$$

To extend the maximum principle to the whole $\mathbb{R}^n$, first of all note that for $t = 0$ there is nothing to prove:

$$v(x,0) \leqslant u(x,0) \leqslant g(x). \tag{1.306}$$

Suppose now $0 < t \leqslant T$. Our goal is to prove that $u(y,t) \leqslant \sup_{z \in \mathbb{R}^n} g(z)$ for all $y \in \mathbb{R}^n$. To prove this we proceed as follows. Fix r, take $x \in \partial B(y,r)$, $|x-y| = r$. Then:

$$\begin{aligned} v(x,t) =& u(x,t) - \frac{\mu}{(T+\epsilon-t)^{\frac{n}{2}}} e^{\frac{|x-y|^2}{4(T+\epsilon-t)}} \leqslant Ae^{a(|y|+r)^2} - \frac{\mu}{(T+\epsilon)^{\frac{n}{2}}} e^{\frac{|x-y|^2}{4(T+\epsilon)}} \leqslant \\ \leqslant& Ae^{a(|y|+r)^2} - \mu(4(a+\gamma))^{n/2} e^{(a+\gamma)r^2}, \qquad 0 < t \leqslant T, \epsilon > 0. \end{aligned} \tag{1.307}$$

where we used that the condition $4a(T+\epsilon) < 1$ implies

$$\frac{1}{4(T+\epsilon)} = a + \gamma > 0, \qquad \gamma > 0.$$

Clearly, Eq. (1.307) implies that there exists r large enough so that $v(x,t) \leqslant \sup_z g(z)$, for all $x \in \partial B(y,r)$, and in particular for $x = y$. Therefore, by the maximum principle for v, $v(x,t) \leqslant \sup_{z \in \mathbb{R}^n} g(z)$ for all $x \in B(y,r)$. Taking the $\mu \to 0$ limit, and using that $\lim_{\mu \to 0} v(y,t) = u(y,t)$, we finally get:

$$u(y,t) \leqslant \sup_{z \in \mathbb{R}^n} g(z)\ , \tag{1.308}$$

which is what we wanted to prove. If $4aT \geqslant 1$, we divide T into small subintervals, and repeat the argument for each subinterval. □

This theorem can be used to prove uniqueness of the solution of the Cauchy problem.

Theorem 1.25. *Let $g \in C(\mathbb{R}^n)$, $f \in C(\mathbb{R}^n \times [0,T])$. Then, there exists at most one solution $u \in C_1^2(\mathbb{R}^n \times [0,T]) \cap C(\mathbb{R}^n \times [0,T])$ of:*

$$\begin{cases} u_t - \Delta u = f & \text{in } \mathbb{R}^n \times (0,T), \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases} \tag{1.309}$$

such that $u(x,t) \leqslant Ae^{a|x|^2}$ for $x \in \mathbb{R}^n$, $0 \leqslant t \leqslant T$ for some $A, a > 0$.

Proof. Let $u, \tilde{u}$ be two solutions. Then, apply the maximum principle for $w = \pm(u - \tilde{u})$. □

1.4.4 Regularity

The next step is to show that, as for the Laplace equation, if u is a solution of the heat equation, then it is automatically smooth.

Theorem 1.26 (Regularity of the solution of the heat equation)**.** *Suppose $u \in C_1^2(U_T)$ is a solution of the heat equation. Then,*

$$u \in C^\infty(U_T). \tag{1.310}$$

Remark 1.29. *The statement is also true if the boundary values of u are non-smooth on Γ_T. The proof is based on the use of the mollifiers as for the corresponding result for the Laplace equation.*

Proof. To begin, let us consider a ball $B(0,r) \subset U_T$. We would like to prove that $u \in C^\infty(B(0,r/4))$. The same argument can be repeated for balls centered in any point in U_T, and this would conclude the proof. Let θ be a smooth, compactly supported function such that

$$\theta = \begin{cases} 1, & \text{if } (x,t) \in B(0,r/2), \\ 0 & \text{if } (x,t) \in B(0,r)^c. \end{cases} \tag{1.311}$$

Suppose that $B(0,r) \subset U_T$. Let

$$v(x,t) = \theta(x,t)u(x,t). \tag{1.312}$$

Then, $v(x,t) \in C_1^2(U_T)$, and v is compactly supported. We compute

$$\partial_t v \quad = \partial_t(\theta u) = (\partial\theta)u + \theta\partial u, \tag{1.313}$$

$$\Delta v \quad = (\Delta\theta)u + \theta\Delta u + 2D\theta \cdot Du. \tag{1.314}$$

Therefore,

$$\begin{aligned} \partial_t v - \Delta v &= \theta\partial tu + \partial_t\theta u - (\Delta\theta)u + \theta\Delta u + 2D\theta \cdot Du = \\ &= \theta\,\underbrace{(\partial_t u - \Delta u)} = 0 + (\partial_t\theta)u - (\Delta\theta)u - 2D\theta \cdot Du \equiv f(x,t), \end{aligned} \tag{1.315}$$

in particular

$$f(x,t) = (\partial_t\theta)u - (\Delta\theta)u - 2D\theta \cdot Du. \tag{1.316}$$

Therefore, being the solution of the heat equation unique among the bounded functions, we can write down v thanks to Duhamel's principle. Let now t_0 be the time of the initial datum, and suppose that $|t_0| > r$. Therefore, $(x, t_0) \neq B(0, r)$ for all x, so $v(x, t_0) = 0$ for all x. By Duhamel's formula,

$$\begin{aligned} v(x,t) &= \int_{t_0}^{t} ds \int_{\mathbb{R}^n} dy \Phi(x-y, t-s) f(y,s) \equiv \\ &\equiv \int_{B(0,r)\backslash B(0,r/2)} dyds \Phi(x-y, t-s) \left[(\partial_t \theta) u - (\Delta\theta) u - 2D\theta \cdot Du\right] \, . \end{aligned} \tag{1.317}$$

Notice that $f = 0$ if $(y, s) \in B(0, r/2)$, being θ constant. Le us now choose $(x, t) \in B(0, r/4)$: $(x - y, t - s) \neq (0, 0)$ since $y \in B \backslash B'$, then $v(x, t)$ is $C^\infty(B(0, r/4))$ since Φ is C^∞ away from $(0, 0)$. □

Therefore, we proved that the solution of the heat equation is unique and bounded in bounded domains, and on unbounded domains we proved that $|u(x,t)| \leqslant Ae^{a|x|^2}$. Also, the solution is $C^\infty(U_T)$. It is however possible to see that there exist infinitely many solutions of the heat equation that do not fulfill the exponential bound. This is the content of the next theorem, whose proof will be omitted.

Proposition 6. *There exists a function $u \in C_1^2(\mathbb{R}^n \times R)$ such that $u_t - \Delta u = 0$, $u(x, 0) = 0$ for all $x \in \mathbb{R}^n$ but $u \neq 0$.*

1.4.5 Long time limit

The next theorem establishes a connection between the infinite time limit of the heat equation and the Laplace equation.

Theorem 1.27 (Convergence to the solution of the Laplace equation)**.** *Let $U \subset \mathbb{R}^n$ open and bounded, ∂U regular and let $g \in C^0(\partial U)$. For every solution $u \in C_1^2(U \times (0, \infty)) \cap C^0(\bar{U} \cap [0, \infty))$ of*

$$\begin{cases} \partial_t u - \Delta u = 0, & \text{in } U \times (0, \infty), \\ u = g, & \text{on } \partial U \times [0, \infty), \end{cases} \tag{1.318}$$

the limit

$$\lim_{t\to\infty} u(\cdot, t) = v, \tag{1.319}$$

exists uniformly in U, with $v \in C^2(U) \cap C^0(\bar{U})$ the solution of the initial value problem

$$\begin{cases} \Delta v = 0, & \text{in } U, \\ v = g, & \text{on } \partial U \, . \end{cases} \tag{1.320}$$

Proof. For $\epsilon > 0$ let us define $w_\epsilon : \mathbb{R}^{n+1} \to \mathbb{R}^n$ as

$$w_\epsilon(x,t) = \cos(\epsilon x_1) e^{-\epsilon^2 t}. \tag{1.321}$$

Then

$$(\partial_t - \Delta) w_\epsilon = (-\epsilon^2 + \epsilon^2) w_\epsilon = 0, \qquad \text{in } \mathbb{R}^{n+1}, \tag{1.322}$$

and $w_\epsilon(x,0) > 0$ for any $x \in [-1/\epsilon, 1/\epsilon]^n$, $w_\epsilon(\cdot,t) \underset{t\to\infty}{\longrightarrow} 0$ uniformly. In particular, we want to take ϵ so small that $U \subset [-1/\epsilon, 1/\epsilon]^n$. We define

$$M = \max_{\bar{U}} \frac{|u(\cdot,0) - v|}{|w_\epsilon|}. \tag{1.323}$$

Let v be the solution of Eq. (1.320). Then $(\partial_t - \Delta)w_\epsilon = 0$, which implies:

$$(\partial_t - \Delta)(u - v - Mw_\epsilon) = 0\ . \tag{1.324}$$

This, together with the inequality

$$u - v - Mw_\epsilon \leqslant 0 \text{ on } \Gamma_\infty\ , \tag{1.325}$$

implies, by the maximum principle:

$$u \leqslant v + Mw_\epsilon \text{ on } U_\infty\ . \tag{1.326}$$

Analogously,

$$u - v + Mw_\epsilon \geqslant 0 \text{ on } \Gamma_\infty, \tag{1.327}$$

which gives, thanks again to the maximum principle,

$$u \geqslant v - Mw_\epsilon \text{ on } U_\infty. \tag{1.328}$$

Thus, $|u - v| \leqslant Mw_\epsilon \underset{t\to\infty}{\longrightarrow} 0$, which proves the claim. □

1.4.6 Energy methods

We conclude the discussion of the heat equation by discussing energy methods.

Theorem 1.28 (Uniqueness of the solution of the heat equation)**.** *There exists at most one solution $u \in C_1^2(\bar{U}_T)$ of (1.258) with $U \subset \mathbb{R}^n$, $\partial U \in C^1$.*

Proof. Let us proceed by contradiction: let us assume that $u, \tilde{u}$ are solutions of (1.258), so $w := u - \tilde{u}$ solves the initial value problem

$$\begin{cases} w_t - \Delta w = 0, & \text{in } U_T, \\ w = 0, & \text{on } \Gamma_T. \end{cases} \tag{1.329}$$

Now set

$$e(t) := \frac{1}{2}\int_U w^2(x,t)dx, \qquad \text{for } 0 \leqslant t \leqslant T. \tag{1.330}$$

We compute:

$$\dot{e}(t) = \int_U w(x,t)\dot{w}(x,t)dx = \int_U w(x,t)\Delta w(x,t)dx = -\int_U |Dw|^2(x,t)dx \leqslant 0, \tag{1.331}$$

meaning that $\frac{de(t)}{dt} \leqslant 0$, that is $e(t) \leqslant e(0) = 0$ for $0 \leqslant t \leqslant T$. This implies $e(t) = 0$ for all $0 \leqslant t \leqslant T \Longrightarrow w(x,t) = 0$ in U_T, *i.e.* $u - \tilde{u} = 0$ in U_T. □

Let us now discuss the backward initial value problem associated to the heat equation.

Theorem 1.29 (Backwards uniqueness)**.** *Let $u, \tilde{u} \in C^2(\bar{U}_T)$ be the solutions of*

$$\begin{cases} u_t - \Delta u = f & \text{in } U_T, \\ u = g & \text{on } \partial U \times [0,T], \end{cases} \qquad \begin{cases} \tilde{u}_t - \Delta \tilde{u} = f & \text{in } U_T, \\ \tilde{u} = g & \text{on } \partial U \times [0,T], \end{cases} \tag{1.332}$$

for some given g, and

$$u(x,T) = \tilde{u}(x,T), \qquad x \in U. \tag{1.333}$$

Then

$$u \equiv \tilde{u} \text{ within } U_T. \tag{1.334}$$

Remark 1.30. *Note that we are not supposing that $u = \tilde{u}$ for $t = 0$.*

Proof. Let $w := u - \tilde{u}$ and

$$e(t) := \frac{1}{2} \int_U w^2(x,t)dx, \qquad 0 \leqslant t \leqslant T\ . \tag{1.335}$$

Then, as in the proof of the previous theorem:

$$\dot{e}(t) = -2 \int_U |Dw|^2(x,t)dx \leqslant 0. \tag{1.336}$$

Furthermore,

$$\ddot{e}(t) = -4 \int_U Dw \cdot Dw_t dx = 4 \int_U \Delta w w_t = 4 \int_U (\Delta w)^2 dx, \tag{1.337}$$

where we just integrated by parts. Being $w = 0$ on ∂U, by the Hölder inequality we get

$$\int_U |Dw|^2 dx = -\int_U w \Delta w dx \leqslant \left(\int_U w^2 dx \right)^{\frac{1}{2}} \left(\int_U (\Delta w)^2 dx \right)^{\frac{1}{2}} \tag{1.338}$$

By (1.336) and (1.337) we get

$$(\dot{e}(t))^2 = 4 \left(\int_U |Dw|^2 \right)^2 \leqslant \left[\left(\int_U w^2 dx \right)^{\frac{1}{2}} \left(\int_U (\Delta w)^2 dx \right)^{\frac{1}{2}} \right] = e(t)\ddot{e}(t), \tag{1.339}$$

meaning that

$$\ddot{e}(t)e(t) \geqslant (\dot{e}(t))^2, \qquad 0 \leqslant t \leqslant T. \tag{1.340}$$

Of course, if $e(t) = 0$ we are done: we are in the case proved in the previous theorem. On the other hand, if $e(t) \neq 0$, there exists an interval $[t_1, t_2] \subset [0,T]$ such that

$$e(t) > 0 \text{ for } t_1 < t < t_2, e(t_2) = 0. \tag{1.341}$$

Now it is convenient to introduce:

$$f(t) := \log e(t) \Rightarrow \ddot{f}(t) = \frac{\ddot{e}(t)}{e(t)} - \frac{\dot{e}^2(t)}{e^2(t)} \geqslant 0, \tag{1.342}$$

thanks to (1.340): this means that f is *convex* on (t_1, t_2). Hence, if $0 < \tau < 1$, $t_1 < t < t_2$,

$$f((1-\tau)t_1 + \tau t) \leqslant (1-\tau) f(t_1) + \tau f(t), \tag{1.343}$$

i.e.

$$e((1-\tau)t_1 + \tau t) \leqslant e(t_1)^{1-\tau} e^{\tau}(t) \Rightarrow 0 \leqslant e((1-\tau)t_1 + \tau t_2) \leqslant e(t_1)^{1-\tau} e^{\tau}(t_2), \tag{1.344}$$

meaning that $e(t) = 0$ for any $t_1 \leqslant t \leqslant t_2$, which is a contradiction. □

1.5 The wave equation

Definition 21 (Wave equation)**.** *The partial differential equation*

$$u_{tt} - \Delta u = 0, \qquad t > 0, x \in U \subset \mathbb{R}^n \tag{1.345}$$

with U open and bounded is called wave equation. Correspondingly,

$$u_{tt} - \Delta u = f, \qquad t > 0, x \in U \subset \mathbb{R}^n \tag{1.346}$$

for some given f is called non-homogeneous wave equation. The unknown is the function $u : \bar{U} \times [0, \infty) \to \mathbb{R}$.

Motivations. The wave equation describes the motion of vibrating systems, such as strings in one dimension, membranes in two dimensions, or elastic solids in three dimensions. Let $u(x, t)$ be the displacement in some direction of the point x at $t \leqslant 0$, and let $V \subset U$ an arbitrary smooth subregion. If we consider without loss of generality the mass density to be equal to one, the acceleration within V is given by

$$\frac{d^2}{dt^2} \int_V u dx = \int_V u_{tt} dx, \tag{1.347}$$

while the net contact force on V is

$$- \int_{\partial V} F \cdot \nu dS. \tag{1.348}$$

From the second law of the dynamics we know that $F = ma = a$ since $m = 1$, therefore

$$\int_V u_{tt} dx = - \int_{\partial V} F \cdot \nu dS. \tag{1.349}$$

Using the Gauss-Green theorem and the fact that V is arbitrary we get

$$u_{tt} = -\text{div} F. \tag{1.350}$$

For ideal elastic bodies, it is reasonable to assume that F is a function of the gradient of u only:

$$u_{tt} + \text{div} F(Du) = 0\ . \tag{1.351}$$

For small deformations around an equilibrium point, it is reasonable to assume that $F(Du) = -aDu$ for some a. Therefore,

$$u_{tt} + \text{div}(-aDu) = u_{tt} - a\Delta u = 0. \tag{1.352}$$

Remark 1.31. *The presence of the second order time derivative implies that, in order to find a solution, we will have to specify $u(x, t = 0)$ and $u_t(x, t = 0)$.*

1.5.1 Solution in one dimension: d'Alembert formula

Consider the initial value/boundary value problem:

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u = g, u_t = h & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases} \tag{1.353}$$

for g and h given. In order to find the solution, we use the following identity:

$$u_{tt} - \Delta u = \left(\frac{\partial}{\partial_t} + \frac{\partial}{\partial_x}\right)\left(\frac{\partial}{\partial_t} - \frac{\partial}{\partial_x}\right) u = u_{tt} - u_{xx} = 0. \tag{1.354}$$

Now, let

$$v(x,t) = \left(\frac{\partial}{\partial_t} v \frac{\partial}{\partial_x}\right) u(x,t), \tag{1.355}$$

we immediately recognize that

$$\left(\frac{\partial}{\partial_t} + \frac{\partial}{\partial_x}\right) v(x,t) = v_t(x,t) + v_x(x,t) = 0, \qquad x \in \mathbb{R}, t > 0, \tag{1.356}$$

is a transport equation with constant coefficients, Eq. (1.14). The solution is:

$$v(x,t) = a(x-t), \qquad a(x) := v(x,0). \tag{1.357}$$

This means that

$$u_t(x,t) - u_x(x,t) = a(x,t), \qquad R \times (0,\infty), \tag{1.358}$$

which is in turn a *non-homogeneous transport equation*, Eq. (1.18). The solution is:

$$u(x,t) = \int_0^t a(x+(t-s)-s)ds + b(x+t) = \frac{1}{2}\int_{x-t}^{x+t} a(y)dy + b(x+t), \qquad b(x) := u_t(x,0). \tag{1.359}$$

Now we are left with fixing a and b, which we shall do using the two boundary conditions. We have:

$$u(x,0) = g(x) \Rightarrow b(x) = g(x), \qquad x \in \mathbb{R}, \tag{1.360}$$

$$u_t(x,0) = h(x) \Rightarrow a(x) = v(x,0) = u_t(x_0) - u_x(x,0) = h(x) - g'(x), \qquad x \in \mathbb{R}. \tag{1.361}$$

Plugging all these informations into the solution, we find

$$\begin{aligned} u(x,t) =& \frac{1}{2}\int_{x-t}^{x+t} (h(y) - g'(y))dy + g(x+t) \Rightarrow \\ \Rightarrow u(x,t) =& \frac{1}{2}\left[g(x+t) + g(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy, \end{aligned} \tag{1.362}$$

which is known as the d'Alembert formula.

Theorem 1.30 (Solution of the wave equation for $n = 1$)**.** *Assume $g \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R})$, and u defined by the d'Alembert formula (1.362). Then*

1. $u \in C^2(\mathbb{R} \times [0,\infty))$,
2. $u_{tt} - u_{xx} = 0$ *in* $\mathbb{R} \times (0,\infty)$,
3. $$\lim_{(x,t)\underset{t>0}{\longrightarrow}(x^0,0)} u(x,t) = g(x^0), \qquad \lim_{(x,t)\underset{t>0}{\longrightarrow}(x^0,0)} u_t(x,t) = h(x^0).$$

Proof. Left to the reader. □

Remark 1.32. 1. *The solution of the wave equation has the form:*

$$u(x,t) = F(x+t) + G(x-y) \tag{1.363}$$

for some F and G. Conversely, only functions of this form solve $u_{tt} - u_{xx} = 0$, meaning that the general solution of the one dimensional wave equation is a linear combination of the solutions of $u_t - u_x = 0$ and $u_t + u_x = 0$.

2. *If $g \in C^k$, $h \in C^{k-1} \Rightarrow u \in C^k$, but not smoother in general. In contrast to the heat equation, the wave equation does not introduce smoothing of the initial datum.*

Example 1.5.1 (Wave equation on the half line)**.** Let us consider the wave equation on the half line $\mathbb{R}_+$:

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}_+ \times (0,\infty), \\ u = g, u_t = h & \text{on } \mathbb{R}_+ \times \{0\}, \\ u = 0 & \text{on } 0 \times (0,\infty) \end{cases} \tag{1.364}$$

with g, h given, such that $g(0) = h(0) = 0$. It is convenient to extend the solution via an odd reflection:

$$\begin{aligned} \tilde{u}(x,t) &= \begin{cases} u(x,t) & \text{if } x \geqslant 0, t \geqslant 0, \\ -u(-x,t) & \text{if } x \leqslant 0, t \geqslant 0, \end{cases} \\ \tilde{g}(x,t) &= \begin{cases} g(x,t) & \text{if } x \geqslant 0, t \geqslant 0, \\ -g(-x,t) & \text{if } x \leqslant 0, t \geqslant 0, \end{cases} \\ \tilde{h}(x,t) &= \begin{cases} h(x,t) & \text{if } x \geqslant 0, t \geqslant 0, \\ -h(-x,t) & \text{if } x \leqslant 0, t \geqslant 0. \end{cases} \end{aligned} \tag{1.365}$$

Being the problem

$$\begin{cases} \tilde{u}_{tt} - \Delta \tilde{u} = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ \tilde{u} = \tilde{g}, \tilde{u}_t = \tilde{h} & \text{on } \mathbb{R}U \times \{0\}, \\ \tilde{u} = 0 & \text{on } 0 \times (0,\infty) \end{cases} \tag{1.366}$$

defined on the whole line, we can use the d'Alembert formula (1.362), thus getting

$$\tilde{u}(x,t) = \frac{1}{2}\left[\tilde{g}(x+t) + \tilde{g}(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} \tilde{h}(y)dy. \tag{1.367}$$

Restricting the solution to the domain $\{x \geqslant 0, t \geqslant 0\}$, we have:

$$u(x,t) = \begin{cases} \frac{1}{2}\left[g(x+t) + g(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy, & \text{if } x \geqslant t \geqslant 0, \\ \frac{1}{2}\left[g(x+t) - g(-t+x)\right] + \frac{1}{2}\int_{-x+t}^{x+t} h(y)dy, & \text{if } 0 \leqslant x \leqslant t. \end{cases} \tag{1.368}$$

If $h \equiv 0$, the last formula shows that the initial displacement g is propagating both in left and in right direction with the same velocity. Finally, the integral of h reflects off the point $x = 0$ where the vibrating string is fixed.

1.5.2 Solution in higher dimensions

For $n \geqslant 2$ the situation is more complicated: the idea is to first find a solution for the *average* of u over certain spheres.

Definition 22. *Let $x \in \mathbb{R}^n$, $t > 0$ and $r > 0$.*

$$U(x; r, t) := \fint_{\partial B(x,r)} u(y,t) dS(y) \tag{1.369}$$

is the average of $u(\cdot, t)$ over ∂B. Similarly

$$G(x; r) = \fint_{\partial B(x,r)} g(y) dS(y, \tag{1.370}$$

$$H(x; r) = \fint_{\partial B(x,r)} h(y) dS(y). \tag{1.371}$$

We shall consider G, H, U as functions of r and t, and see what equation they solve.

Lemma 1.3 (Euler-Poisson-Darboux equation)**.** *Fin $x \in \mathbb{R}^n$, and suppose that u solves*

$$\begin{cases} u_{tt} - \Delta u = 0, & \text{in } \mathbb{R}^n \times (0, \infty), \\ u = g, u_t = h & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases} \tag{1.372}$$

with $u \in C^m(\mathbb{R} \times [0, \infty))$. Then, $U \in C^m(\bar{\mathbb{R}} \times [0, \infty))$ and

$$\begin{cases} U_{tt} - U_{rr} - \frac{n-1}{r} U_r = 0, & \text{in } \mathbb{R}_+ \times (0, \infty), \\ U = G, U_t = H, & \text{on } \mathbb{R}_+ \times \{t = 0\}. \end{cases} \tag{1.373}$$

Proof.

$$\partial_r U(x; r, t) = \partial_r \fint_{\partial B(x,r)} u(y,t) dS(y) = \partial_r \fint_{\partial B(0,1)} u(x + rz, t) dS(z) = \frac{r}{n} \fint_{B(x,r)} \Delta u(y,t) dy \tag{1.374}$$

Therefore, since Δu is continuous,

$$\lim_{r \to 0^+} \partial_r U(x; r, t) = 0. \tag{1.375}$$

Let us now compute the second derivative

$$\begin{aligned} \partial_r^2 U(x; r, t) &= \partial_r \frac{1}{n\alpha(n) r^{n-1}} \int_{B(x,r)} \Delta u(y,t) dy = \\ &= \frac{1-n}{r} \frac{1}{n\alpha(n) r^{n-1}} \int_{B(x,r)} \Delta u(y,t) dy + \frac{1}{n\alpha(n) r^{n-1}} \underbrace{\partial_r \int_{B(x,r)} \Delta u(y,t) dy}_{\int_{\partial B(x,r)} \Delta u(y,t) dy}, \end{aligned} \tag{1.376}$$

that is

$$\begin{aligned} \partial_r^2 U(x; r, t) &= \frac{1-n}{r} \partial_r U(x; r, t) + \fint_{\partial B(x,t)} \Delta u(y,t) dy = \\ &= \frac{1-n}{r} \fint_{B(x,r)} \Delta u dy + \fint_{\partial B(x,r)} \Delta u dS \end{aligned} \tag{1.377}$$

so that

$$\lim_{r\to 0^+} \partial_r^2 U(x;r,t) = \frac{1-n}{n}\Delta u(x,t) + \Delta u(x,t) = \frac{1}{n}\Delta u(x,t). \tag{1.378}$$

These formula show that $U \in C^2$. One can also compute the higher derivatives, and check that $U \in C^m$ if $u \in C^m$. Let us now check that U solves the EPD equation. We have:

$$\begin{aligned} U_r &= \frac{r}{n} \fint_{B(x,r)} u_{tt} dy \Rightarrow r^{n-1} U_r = \frac{1}{n\alpha(n)} \int_{B(x,r)} u_{tt} dy \\ \Rightarrow \left(r^{n-1}U_r\right)_r &= \frac{1}{n\alpha(n)} \int_{\partial B(x,r)} u_{tt} dS = r^{n-1} \fint_{\partial B(x,r)} u_{tt} ds = r^{n-1} U_{tt}, \end{aligned} \tag{1.379}$$

which proves the claim since

$$\left(r^{n-1}U_r\right)_r = \frac{n-1}{r} r^{n-1} U_r + r^{n-1} U_{rr} \Rightarrow U_{tt} - \frac{n-1}{r} U_r - U_{rr} = 0. \tag{1.380}$$

□

Remark 1.33. $U_{rr} + \frac{n-1}{r} U_r$ *is the radial part of the Laplacian in spherical coordinates.*

Three dimensional case

So far we discussed the case of general n. Now the main goal will be to reduce the EDP equation to the one-dimensional wave equation. We will first discuss the case $n = 3$. Suppose that $u \in C^2(\mathbb{R}^3 \times [0,\infty))$ solves

$$\begin{cases} u_{tt} - \Delta u = 0, & \text{in } \mathbb{R}^3 \times (0,\infty), \\ u = g, u_t = h, & \text{on } \mathbb{R}^3 \times \{t = 0\}, \end{cases} \tag{1.381}$$

and define

$$\tilde{U} := rU, \qquad \tilde{G} := rG, \qquad \tilde{H} := rH. \tag{1.382}$$

Proposition 7.

$$\begin{cases} \tilde{U}_{tt} - \tilde{U}_{rr} = 0, & \textit{in } \mathbb{R}_+ \times (0,\infty), \\ \tilde{U} = \tilde{G}, \tilde{U}_t = \tilde{H}, & \textit{on } \mathbb{R}_+ \times \{t = 0\}, \\ \tilde{U} = 0, & \textit{on } \{r = 0\} \times (0,\infty). \end{cases} \tag{1.383}$$

Proof. By definition

$$\tilde{U}_{tt} = rU_{tt} = \left(U_{rr} + \frac{2}{r}U_r\right) = rU_{rr} + 2U_r = \left(\underbrace{U + rU_r}_{\tilde{U}_r}\right)_r = \tilde{U}_{rr}. \tag{1.384}$$

Also, $\lim_{r\to 0^+} \tilde{U}(x;r,t) = 0$ for any t. □

Therefore, by d'Alembert's formula (1.362)

$$\tilde{U}(x;r,t) = \frac{1}{2}[\tilde{G}(r+t) - \tilde{G}(t-r)] + \frac{1}{2}\int_{-r+t}^{r+t} \tilde{H}(y)dy\ . \tag{1.385}$$

We can use this explicit expression to find $u(x,t)$. In fact:

$$\begin{aligned} u(x,t) &= \lim_{r\to 0^+} \frac{\tilde{U}(x;t,r)}{r} = \\ &= \lim_{r\to 0^+} \left[\frac{1}{2}\left[\frac{\tilde{G}(r+t)-\tilde{G}(t-r)}{r}\right] + \frac{1}{2r}\int_{-r+t}^{r+t} \tilde{H}(y)dy \right] = \tilde{G}'(t) + \tilde{H}(t), \end{aligned} \tag{1.386}$$

which gives

$$\begin{aligned} u(x,t) =&\partial_t \left(t\fint_{\partial B(x,t)} gdS \right) + t\fint_{\partial B(x,t)} hdS = \\ =&\fint_{\partial B(x,t)} gdS + \underbrace{t\partial_t \fint_{\partial B(x,t)} gdS}_{t\fint_{\partial B(x,t)} Dg(y)\cdot\frac{y-x}{t}dS(y)} + t\fint_{\partial B(x,t)} hdS \end{aligned} \tag{1.387}$$

Thus, we found the solution $u(x,t)$ in terms of the initial data g and h. Eq . (1.387) is known as *Kirchoff formula* for the solution of the three-dimensional wave equation,

$$u(x,t) = \fint_{\partial B(x,t)} \left(h(y) + g(y) + (y-x)\cdot Dg(y)\right) dS(y). \tag{1.388}$$

Two dimensional case

In order to solve the two-dimensional wave equation, we shall look at it as the restriction on a plane of the three-dimensional case. Suppose that $u \in C^2(\mathbb{R}^2 \times [0,\infty))$ solves

$$\begin{cases} u_{tt} - \Delta u = 0, & \text{in } \mathbb{R}^2 \times (0,\infty), \\ u = g, u_t = h, & \text{on } \mathbb{R}^2 \times \{t=0\}. \end{cases} \tag{1.389}$$

Let $\bar{u}(x_1,x_2,x_3,t) := u(x_1,x_2,t)$. Then,

$$\begin{cases} \bar{u}_{tt} - \Delta \bar{u} = 0, & \text{in } \mathbb{R}^3 \times (0,\infty), \\ \bar{u} = \bar{g}, \bar{u}_t = \bar{h} & \text{on } \mathbb{R}^3 \times \{t=0\}, \end{cases} \tag{1.390}$$

where $\bar{g}(x_1,x_2,x_3) = g(x_1,x_2)$ and $\bar{h}(x_1,x_2,x_3) = h(x_1,x_2)$. Thus, by Kirchoff formula (1.388), setting $\bar{x} = (x_1,x_2,0)$:

$$u(x,t) = \bar{u}(\bar{x},t) = \partial_t \left(t\fint_{\partial \bar{B}(\bar{x},t)} \bar{g}d\bar{S} \right) + t\fint_{\partial \bar{B}(\bar{x},t)} \bar{h}d\bar{S}. \tag{1.391}$$

An explicit computation shows that

$$\fint_{\partial \bar{B}(\bar{x},t)} \bar{g}d\bar{S} = \frac{1}{4\pi t^2}\int_{\partial \bar{B}(\bar{x},t)} \bar{g}d\bar{S} = \frac{1}{4\pi t^2}\frac{t}{2}\fint_{B(x,t)} g(y)\frac{1}{\sqrt{t^2-(y-x)^2}}dy, \tag{1.392}$$

and a similar identity holds true for the last term in Eq. (1.391). Therefore,

$$u(x,t) = \frac{1}{2}\frac{\partial}{\partial t}\left(t^2 \fint_{B(x,t)} \frac{g(y)}{\sqrt{t^2-|y-x|^2}}dy\right) + \frac{t^2}{2}\fint_{B(x,t)} \frac{h(y)}{\sqrt{t^2-|y-x|^2}}dy. \tag{1.393}$$

Thus, using

$$t^2 \fint_{B(x,t)} \underbrace{\frac{g(y)}{\sqrt{t^2-|y-x|^2}}}_{\frac{g(y)}{t\sqrt{1-|y-x|^2/t^2}}}dy = t\fint_{B(0,1)} \frac{g(x+tz)}{\sqrt{1-|z|^2}}dz, \tag{1.394}$$

with $z = |y-x|^2/t^2$, we get

$$\begin{aligned} \frac{\partial}{\partial t}\left(t^2 \fint_{B(x,t)} \frac{g(y)}{\sqrt{t^2-|y-x|^2}}dy\right) &= \fint_{B(0,1)} \frac{g(x+tz)}{\sqrt{1-|z|^2}}dz + t\fint_{B(0,1)} \frac{Dg(x+tz)\cdot t}{\sqrt{1-|z|^2}}dz = \\ &= \fint_{B(x,t)} \frac{g(x+tz)}{\sqrt{t^2-|y-x|^2}}dz + t\fint_{B(x,t)} \frac{Dg(y)\cdot(y-x)}{\sqrt{t^2-|y-x|^2}}dz, \end{aligned} \tag{1.395}$$

which finally gives

$$u(x,t) = \frac{1}{2}\fint_{B(x,t)} \frac{tg(y)+t^2h(y)+tDg(y)\cdot(y-x)}{\sqrt{t^2-|y-x|^2}}dy, \tag{1.396}$$

for all $x \in \mathbb{R}^2$, $t > 0$. This is called the Poisson formula for the solution of the wave equation in $n = 2$.

Remark 1.34. *1. The trick of getting the solution for $n = 2$ starting from $n = 3$ is called method of descent.*

2. *The main difference between the solutions in two and three dimensions is that, for $n = 3$ the solution (1.388) at (x,t) only depends on the values of h, g at $|y-x| = t$, while for $n \leqslant 2$ it depends on $|y-x| \leqslant t$ (Huygens principle).*

3. *In contrast to the one-dimensional case, the solution for $n = 2, 3$ involves derivatives of g. Therefore, the solution might not be as regular as the initial datum (loss of regularity).*

To conclude, we give the general expressions for the solution of the wave equation in n dimensions. We refer the reader to [Evans] for the proof.

Theorem 1.31. *Let $n \geqslant 3$ odd, and suppose that $g \in C^{m+1}(\mathbb{R}^n)$, $h \in C^m(\mathbb{R}^n)$ for $m = \frac{n+1}{2}$. Then, the solution of the wave equation is:*

$$u(x,t) = \frac{1}{\gamma_n}\left[\partial_t\left(\frac{1}{t}\partial_t\right)^{\frac{n-3}{2}}\left(t^{n-2}\fint_{\partial B(x,t)} g ds\right) + \left(\frac{1}{t}\partial_t\right)^{\frac{n-3}{2}}\left(t^{n-2}\fint_{\partial B(x,t)} h ds\right)\right], \tag{1.397}$$

where $\gamma_n = 1\cdot 3\cdot 5\cdot\dots\cdot(n-2)$.

Let $n \geqslant 2$ even, and suppose that $g \in C^{m+1}(\mathbb{R}^n)$, $h \in C^m(\mathbb{R}^n)$ for $m = \frac{n+2}{2}$. Then, the solution of the wave equation is:

$$u(x,t) = \frac{1}{\gamma_n}\left[\partial_t\left(\frac{1}{t}\partial_t\right)^{\frac{n-2}{2}}\left(t^n \fint_{B(x,t)} \frac{g(y)}{\sqrt{t^2-|y-x|^2}}dy\right) + \right. \\ \left. + \left(\frac{1}{t}\partial_t\right)^{\frac{n-2}{2}}\left(t^n \fint_{B(x,t)} \frac{h(y)}{\sqrt{t^2-|y-x|^2}}dy\right)\right] \tag{1.398}$$

with $\gamma_n = 2 \cdot 4 \cdot \dots \cdot (n-2) \cdot n$

Remark 1.35. *As for $n = 2, 3$, the solution for odd n only depends on the values of g and h on $\partial B(x,t)$, while the solution of the wave equation for even n depends on the values of g and h on $B(x,t)$.*

1.5.3 Non-homogeneous wave equation

Let us now consider the non-homogeneous problem

$$\begin{cases} u_{tt} - \Delta u = f, & \text{in } \mathbb{R}^n \times (0,\infty), \\ u = 0, u_t = 0 & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{1.399}$$

As for the heat equation, we will solve the equation using the Duhamel principle. Consider the homogeneous problem

$$\begin{cases} u_{tt}(\cdot;s) - \Delta u(\cdot;s) = 0, & \text{in } \mathbb{R}^n \times (s,\infty), \\ u(\cdot;s) = 0, u_t(\cdot;s) = f(\cdot;s) & \text{on } \mathbb{R}^n \times \{t = s\}, \end{cases} \tag{1.400}$$

and define

$$u(x,t) = \int_0^t u(x,t;s)ds, \qquad x \in \mathbb{R}^n, t \geqslant 0. \tag{1.401}$$

Theorem 1.32. *Let $n \geqslant 2$, $f \in C^{[n/2]+1}(\mathbb{R}^n \times [0,\infty))$, and define u as in (1.401). Then, u solves the non-homogeneous initial value problem*

Proof. The proof is a direct computation, and it is left as an exercise to the reader. □

Example 1.5.2 (One dimensional non-homogeneous problem)**.** Recall the d'Alembert formula (1.362). We have

$$u(x,t;s) = \frac{1}{2}\int_{x-t+s}^{x+t-s} f(y,s)dy, \tag{1.402}$$

which implies

$$u(x,t) = \frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} f(y,s)dyds = \frac{1}{2}\int_0^t \int_{x-s}^{x+s} f(y,t-s)dy. \tag{1.403}$$

Example 1.5.3 (Three dimensional non-homogeneous problem)**.** Recall Kirchoff formula (1.388). We get

$$u(x,t;s) = (t-s) \fint_{\partial B(x,t-s)} f(y,s)dS \tag{1.404}$$

which implies:

$$\begin{aligned} u(x,t) = \int_0^t ds (t-s) \fint_{\partial B(x,t-s)} f(y,s)dS = \frac{1}{4\pi}\int_0^t ds \int_{\partial B(x,t-s)} \frac{f(y,s)}{t-s}dS = \\ = \frac{1}{4\pi}\int_0^t ds \int_{\partial B(x,r)} \frac{f(y,t-r)}{r}dr = \frac{1}{4\pi}\int_{B(x,t)} \frac{f(y,t-|y-x|)}{|y-x|}dr, \end{aligned} \tag{1.405}$$

for $x \in \mathbb{R}^3, t \geqslant 0$.

1.5.4 Energy methods

To conclude the discussion of the wave equation, we shall introduce energy methods, as for the Laplace and heat equations. We shall use this method to prove uniqueness of the solution in bounded domains, where no explicit formula is available.

Definition 23 (Energy of the solution of the wave equation)**.** *Let $U \in \mathbb{R}^n$ bounded and open. Let u be the solution of:*

$$\begin{cases} u_{tt} - \Delta u = f, & \textit{in } U_T, \\ u = g & \textit{on } \Gamma_T, \\ u_t = h & \textit{on } U \times \{t=0\}. \end{cases} \tag{1.406}$$

The energy of the solution of the wave equation is defined as:

$$E(t) := \frac{1}{2}\int_U \left(u_t^2(x,t) + |Du(x,t)|^2\right) dx \,. \tag{1.407}$$

It is easy to see that the energy is a conserved quantity:

$$\frac{d}{dt}E(t) = \int_U (u_t u_{tt} + Du \cdot Du_t)dx = \int_U u_t(u_{tt} - \Delta u)dx = 0. \tag{1.408}$$

Theorem 1.33. *There exists at most one function $u \in C^2(\bar{U}_T)$ solving (1.406).*

Proof. Suppose by contradiction that $\tilde{u}$ is another solution, and define $w = u - \tilde{u}$. By definition w solves

$$\begin{cases} w_{tt} - \Delta u = f, & \text{in } U_T, \\ w = 0 & \text{on } \Gamma_T, \\ w_t = 0 & \text{on } U \times \{t=0\}. \end{cases} \tag{1.409}$$

By energy conservation we have:

$$E(t) = E(0) = \int_U \left(w_t(x,0) + |Dw(x_0)|^2\right) dx = 0 \tag{1.410}$$

since $w(t,0) = 0$ and $Dw(x,0) = 0$ for any $x \in U$. Therefore,

$$\int_U w_t^2(x,t) = 0 = \int_U |Du(x,t)|^2, \tag{1.411}$$

that is w is constant in x, t, that implies $w = 0$ in U_T. □

Remark 1.36. *Even though the regularity of the solution might deteriorate in time, the energy is constant in time.*

To conclude, we shall discuss the conservation of energy to further characterize the finite speed of propagation of the wave equation. Suppose $u \in C^2$ solves the wave equation $u_{tt} - \Delta u = 0$ in $\mathbb{R}^n \times (0, \infty)$

Definition 24 (Backwards wave cone)**.** *The backwards wave cone with apex (x_0, t_0) is*

$$K(x_0,t_0) = \left\{(x,t)\middle|0 \leqslant t \leqslant t_0, |x - x_0| \leqslant t_0 - t\right\}. \tag{1.412}$$

Theorem 1.34. *if $u = u_t = 0$ on $B(x_0,t_0) \times \{t = 0\}$, then $u = 0$ within $K(x_0,t_0)$.*

Remark 1.37. *The latter theorem tells us that the solution at given time does not depend on what happens outside $K(x_0,t_0)$: in other words, suppose that $u, \tilde{u}$ are two solutions, with initial data $g, \tilde{g}, h, \tilde{h}$. Suppose that $g = \tilde{g}, h = \tilde{h}$ in $B(x_0,t_0)$. Then, $u(x_0,t_0) = \tilde{u}(x_0,t_0)$. Moreover, since $K(x_0', t_0') \subset K(x_0,t_0)$ if $t_0' \leqslant t_0$, $B(x_0', t_0') \subset B(x_0,t_0)$, we also have $u(x,t) = \tilde{u}(x,t)$ for all $x, t \in K(x_0,t_0)$.*

Proof. Let us define the local energy (*energy of a given "slice" of the cone*) as

$$e(t) = \frac{1}{2}\int_{B(x_0,t_0-t)} \left(u_t^2(x,t) + |Du(x,t)|^2\right) dx, \qquad 0 \leqslant t \leqslant t_0. \tag{1.413}$$

A direct computation gives:

$$\begin{aligned}
\frac{d}{dt}e(t) &= \int_{B(x_0,t_0-t)} (u_t u_{tt} + Du \cdot Du_t)\, dx - \frac{1}{2}\int_{\partial B(x_0,t_0-t)} \left(u_t^2 + |Du|^2\right) dx = \\
&= \int_{B(x_0,t_0-t)} u_t(u_{tt} - \Delta u) dx = \\
&= \int_{\partial B(x_0,t_0-t)} \nu \cdot Du u_t dS - \frac{1}{2}\int_{\partial B(x_0,t_0-t)} u_t^2 + |Du|^2 dS = \\
&= \int_{\partial B(x_0,t_0-t)} \left(\nu \cdot Du u_t - \frac{1}{2}u_t^2 - \frac{1}{2}|Du|^2\right) dS.
\end{aligned} \tag{1.414}$$

□

Now, we use Cauchy-Schwarz inequality to bound

$$|\nu \cdot Du u_t| \leqslant |Du||u_t| \leqslant \frac{1}{2}u_t^2 + \frac{1}{2}|Du|^2. \tag{1.415}$$

Therefore,

$$\frac{d}{dt}e(t) \leqslant 0 \Rightarrow e(t) \leqslant e(0) = 0, \tag{1.416}$$

which implies that $u_t = 0$ and $Du = 0$ in $B(x_0, t_0 - t)$ for all $t \leqslant t_0$, that is $u(x,t)$ is constant in $K(x_0, t_0)$. Combined with the vanishing of the initial datum, this implies that $u(x,t) = 0$ in $K(x_0, t_0)$.

Remark 1.38. *Energy conservation allows to prove a similar result for more general partial differential equations, e.g. the nonlinear wave equation.*

Chapter 2

The Fourier transform

2.1 Elements of the theory of L^p spaces

Definition 25 (L^p spaces)**.** *Let $p \in \mathbb{R}$, $1 \leqslant p < \infty$. We define:*

$$L^p(\mathbb{R}^n) := \left\{ f \mid f : \mathbb{R}^n \to \mathbb{C},\ f \text{ measurable}, \int dx\, |f(x)|^p < \infty \right\} . \tag{2.1}$$

Remark 2.1. *The integral $\int dx \cdots$ is a Lebesgue integral. If the function f is Riemann integrable, then it coincides with the standard Riemann integral. More generally one could replace "dx" by a more general measure "$\mu(dx)$" and $\mathbb{R}^n$ by a measurable set $\Omega \in \mathbb{R}^n$:*

$$L^p(\mathbb{R}^n) \longrightarrow L^p(\Omega, d\mu).$$

One can check that L^p is a vector space.

Definition 26 (L^p norm)**.** *For each $f \in L^p$, we define the L^p norm of f as*

$$\|f\|_p \equiv \|f\|_{L^p(\mathbb{R}^n)} := \left(\int dx\, |f(x)|^p \right)^{1/p} . \tag{2.2}$$

Remark 2.2. $\|\cdot\|_p$ *has the following properties:*

1. $\|\lambda f\|_p = |\lambda| \|f\|_p$, $\lambda \in \mathbb{C}$.
2. $\|f\|_p = 0 \Leftrightarrow f(x) = 0$ *a.e.*
3. $\|f + g\|_p \leqslant \|f\|_p + \|g\|_p$

Actually, these properties only imply that $\|\cdot\|_p$ is a semi-norm. It is easy to imagine functions such that $\|f\|_p = 0$ and $f(x) \neq 0$ (take f to be zero everywhere except at an isolated point). To ensure that $\|\cdot\|$ defines a norm, one has to redefine L^p by identifying functions that differ on a zero measure set (e.g., on a countable set of points).

Definition 27 (Re-definition of L^p spaces). *Given $f \in L^p$, we define an equivalent class of functions as*

$$\tilde{f} = \{f' \in L^p \mid f - f' = 0 \ a.e.\} \tag{2.3}$$

We redefine L^p as the set of the equivalence classes of functions $\tilde{f}$.

The L^∞ space is defined as follows.

Definition 28 (L^∞ and $\|\cdot\|_\infty$).

$$L^\infty(\mathbb{R}^n) := \left\{ f \middle| f : \mathbb{R}^n \to \mathbb{C}, \ , f \ measurable \ , \exists K > 0 \ s.t. \ |f(x)| \leqslant K \ a.e. \ \right\}. \tag{2.4}$$

For $f \in L^\infty(\mathbb{R}^n)$ we define as the essential supremum of f:

$$\|f\|_\infty \equiv \|f\|_{L^\infty(\mathbb{R}^n)} := \inf\{K \mid |f(x)| \leqslant K \ a.e. \ in \ \mathbb{R}^n\}. \tag{2.5}$$

We will not go through the theory of L^p spaces; we refer the reader to [Lieb-Loss]. Instead, we shall only recall some some selected results, that will be used later on.

Theorem 2.1 (Completeness). *Let $1 \leqslant p \leqslant \infty$, and let f^i, $i = 1, 2, 3, \cdots$ be a Cauchy sequence in $L^p(\mathbb{R}^n)$:*

$$\|f^i - f^j\|_p \underset{i,j\to\infty}{\longrightarrow} 0. \tag{2.6}$$

Then, there exists $f_* \in L^p$ such that

$$\lim_{i\to\infty} \|f_i - f_*\|_p = 0 . \tag{2.7}$$

Remark 2.3. *We use the notation $f_i \underset{i\to\infty}{\longrightarrow} f$ and we say that f^i converges strongly to f.*

Theorem 2.2 (Approximation by C_c^∞ functions). *Let $f \in L^p(\mathbb{R}^n)$, $1 \leqslant p < \infty$. Then, there exists a sequence of functions $\{f^i\}_{i\in\mathbb{N}}$, $f^i \in C_c^\infty(\mathbb{R}^n)$ such that $f^i \longrightarrow f$ in L^p.*

Remark 2.4. *That is, the smooth compactly supported functions are dense in L^p.*

2.2 The Fourier transform of L^1 functions

We are now ready to introduce the Fourier transform for L^1 functions.

Definition 29 (Fourier transform for L^1 functions). *Let $u \in L^1(\mathbb{R}^n)$. We define its Fourier transform $\hat{u}$ as*

$$\hat{u}(k) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int dx e^{-ik\cdot x} u(x), \qquad k \in \mathbb{R}^n. \tag{2.8}$$

Remark 2.5. *Since $|e^{-ik\cdot x}| = 1$ and $u \in L^1(\mathbb{R}^n)$, $\hat{u}$ is well defined.*

Let us summarize some important properties of the Fourier transform.

Lemma 2.1.
1. *The map $u \to \hat{u}$ is linear in u.*
2. *Let τ_h be the shift operator: $(\tau_h f)(x) = f(x+h)$. Then $\widehat{(\tau_h f)}(k) = e^{-ik\cdot x} \hat{f}(k)$*

3. *Let δ_λ be the scaling operator: $(\delta_\lambda f)(x) = f(x/\lambda)$. Then, $\widehat{(\delta_\lambda f)}(k) = \lambda^n \hat{f}(\lambda k)$*

Proof. Left to the reader. □

Lemma 2.2. 1. *$|\hat{u}(k)| \leqslant \|u\|_1$, $u \in \mathbb{R}^n$.*

2. *$k \to \hat{u}(k)$ is continuous*

Proof. 1. Obvious.

2.
$$\hat{u}(k) - \hat{u}(0) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int (e^{-ik\cdot x} - 1) f(x). \tag{2.9}$$

Since $(e^{-ik\cdot x} - 1)f(x) \underset{|k|\to 0}{\longrightarrow} 0$ and $|e^{-ik\cdot x} - 1||f(x)| \leqslant 2|f(x)| \in L^1$, dominated convergence implies:

$$\lim_{k\to 0} |\hat{u}(k) - \hat{u}(0)| = 0.$$

□

Lemma 2.3 (Fourier transform of convolutions)**.** *Let $f, g \in L^1(\mathbb{R}^n)$. Consider the convolution*

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y) g(y) dy. \tag{2.10}$$

Then
$$\widehat{f * g}(k) = \hat{f}(k)\hat{g}(k)(2\pi)^{\frac{n}{2}} \ . \tag{2.11}$$

Proof. To begin, let us prove that the convolution is in $L^1(\mathbb{R}^n)$. We have:

$$\int dx |f * g|(x) \leqslant \int dx dy |f(y)||g(x-y)| = \int dy |f(y)| \int dx |g(x-y)| = \|f\|_1 \|g\|_1 \tag{2.12}$$

where the last step follows from Fubini's theorem. Then,

$$\begin{aligned}\widehat{(f * g)}(k) =& \frac{1}{(2\pi)^{\frac{n}{2}}} \int dx e^{-ik\cdot x} (f * g)(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int dx dy e^{-ik\cdot x} f(x-y) g(y) = \\ =& (2\pi)^{\frac{n}{2}} \frac{1}{(2\pi)^{\frac{n}{2}}} \int dx dy e^{-ik\cdot(x-y)} e^{-ik\cdot y} f(x-y) g(y) = (2\pi)^{\frac{n}{2}} \hat{f}(k)\hat{g}(k) \ .\end{aligned} \tag{2.13}$$

□

2.3 The Fourier transform of L^2 functions

It is natural to ask whether the Fourier transform can be extended to functions that are not in L^1. For these functions, the definition (2.8) does not make sense a priori (the integral might be infinite). In particular, we will develop the theory of the Fourier transform for L^2 functions. As a preliminary result, let us compute the Fourier transform of the Gaussian.

Theorem 2.3. *Let $\lambda > 0$, and let $g_\lambda(x) = \exp\big(-\lambda\frac{|x|^2}{2}\big)$ be the Gaussian function. Then*

$$\hat{g}_\lambda(k) = \lambda^{-\frac{n}{2}} \exp\left(-\frac{|k|^2}{2\lambda}\right) . \tag{2.14}$$

Proof. By scaling, it is enough to consider the case $\lambda = 1$. Also, since $g_1(x) = \prod_{i=1}^n \exp\left(-\frac{x_i^2}{2}\right)$, it is enough to consider the case $n = 1$. We have:

$$\hat{g}_1(k) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int dx e^{-ik\cdot x} e^{-\frac{x^2}{2}} = \frac{1}{(2\pi)^{\frac{1}{2}}} \int dx e^{-\frac{(x+iy)^2}{2} - \frac{k^2}{2}} \equiv g_1(k) f(k), \tag{2.15}$$

where we defined $f(k) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int dx e^{-\frac{(x+iy)^2}{2}}$. By dominated convergence, we can differentiate under the integral sign:

$$\frac{d}{dk} f(k) = \int_{\mathbb{R}} \frac{dx}{(2\pi)^{\frac{1}{2}}} (-(x+ik)) i e^{-\frac{(x+ik)^2}{2}} = \int_{\mathbb{R}} \frac{dx}{(2\pi)^{\frac{1}{2}}} i \frac{d}{dx} e^{-\frac{(x+ik)^2}{2}} = 0. \tag{2.16}$$

This means that $f(k)$ is a constant and, in particular, $f(k) = f(0) = 1$. □

The key result in order to extend the Fourier transform to L^2 functions is Plancherel's theorem.

Theorem 2.4 (Plancherel)**.** *Let $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then, $\hat{u} \in L^2(\mathbb{R}^n)$ and $\|\hat{u}\|_2 = \|u\|_2$.*

Proof. 1. Let $v, w \in L^1(\mathbb{R}^n)$, Then $\hat{v}, \hat{w} \in L^\infty(\mathbb{R}^n)$. Moreover,

$$\int_{\mathbb{R}^n} dx v(x) \hat{w}(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} dy dx v(x) e^{-ix\cdot y} w(y) = \int_{\mathbb{R}^n} \hat{v}(x) w(x). \tag{2.17}$$

Recalling the Fourier transform of the Gaussian, we have that

$$\frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} dx e^{-ik\cdot x} e^{-\epsilon|x|^2} \equiv g_{2\epsilon}(k) e^{-\frac{k^2}{4\epsilon}} = \frac{1}{(2\epsilon)^{\frac{n}{2}}} e^{-\frac{k^2}{4\epsilon}}. \tag{2.18}$$

Let us now consider the identity (2.17) with $v = g_{2\epsilon}$:

$$\int_{\mathbb{R}^n} \hat{w}(k) e^{-\epsilon|k|^2} dk = \int_{\mathbb{R}^n} dx w(x) e^{-\frac{|x|^2}{4\epsilon}} \frac{1}{(2\epsilon)^{\frac{n}{2}}}. \tag{2.19}$$

In particular, notice that $g_{2\epsilon}/(2\pi)^{\frac{n}{2}}$ is a mollifier, in the sense discussed when proving smoothness of solutions of Laplace equation (even though it is not compactly supported, but this does not change much). Therefore, if w is continuous at $x = 0$

$$\lim_{\epsilon\to 0} \frac{1}{(2\epsilon)^{\frac{n}{2}}} \int dx w(x) e^{-\frac{|x|^2}{4\epsilon}} = w(0)(2\pi)^{\frac{n}{2}} \tag{2.20}$$

2. Let $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, and let $v(x) = \bar{u}(-x)$. We have $w = u * v \Rightarrow \hat{w}(k) = (2\pi)^{\frac{n}{2}} \hat{u}(k)\hat{v}(k)$, where

$$\hat{v}(k) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ik\cdot x} \bar{u}(-x) dx \equiv \bar{\hat{u}}(k) \Rightarrow \hat{w}(k) = (2\pi)^{\frac{n}{2}} |\hat{u}|^2. \tag{2.21}$$

By dominated convergence, w is continuous at zero. In fact:

$$|w(x) - w(0)| \leqslant \int dy|u(x-y)v(y) - u(-y)v(y-x))| = \int dy|u(x-y) - u(-y)||v(y)| \quad (2.22)$$

The argument of the integral is integrable uniformly in x:

$$\begin{aligned} \int dy|u(x-y) - u(-y)||v(y)| &\leqslant \left(\int dy(|u(x-y)| + |u(y)|)\right)^{\frac{1}{2}} \left(\int dy|v(y)|^2\right)^{\frac{1}{2}} \leqslant \\ &\leqslant \left(2\left(\|u\|_2^2 + \|u\|_2^2\right)\right)^{\frac{1}{2}} \|v\|_2^2 \leqslant C. \end{aligned} \quad (2.23)$$

Therefore, by dominated convergence:

$$\lim_{|x|\to 0} |w(x) - w(0)| = \lim_{|x|\to 0} \int dy v(y)(u(x-y) - u(-y)) = 0 \quad (2.24)$$

Hence, by Eq. (2.19):

$$\lim_{\epsilon\to 0} \frac{1}{(2\epsilon)^{\frac{n}{2}}} \int dx w(x) e^{-\frac{|x|^2}{4\epsilon}} = \lim_{\epsilon\to 0} \int dx \hat{w}(x) e^{-\epsilon|x|^2} = w(0)(2\pi)^{\frac{n}{2}} \equiv \|u\|_2^2 (2\pi)^{\frac{n}{2}} . \quad (2.25)$$

By monotone convergence,

$$\lim_{\epsilon\to 0} \int_{\mathbb{R}^n} (2\pi)^{\frac{n}{2}} |\hat{u}|^2 e^{-\epsilon|x|^2} = \int_{\mathbb{R}^n} (2\pi)^{\frac{n}{2}} |\hat{u}|^2 \equiv (2\pi)^{\frac{n}{2}} \|\hat{u}\|_2^2 , \quad (2.26)$$

which proves that $\hat{u} \in L^1 \cap L^2$. Moreover, Eq. (2.25) implies that $\|u\|_2 = \|\hat{u}\|_2$, which concludes the proof.

□

We are now ready to define the Fourier transform in L^2. We shall use an approximation argument. Let $u \in L^2(\mathbb{R}^n)$. Recall that $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ and, in particular, that $C_c^\infty(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Let $\{u_j\}_{j\in\mathbb{N}}$, $u_j \in C_c^\infty(\mathbb{R}^n)$, be an approximation sequence for u, that is $u_j \to u$ in L^2. By Plancherel:

$$\|\hat{u}_i - \hat{u}_j\|_2 = \|u_i - u_j\|_2 , \quad (2.27)$$

which implies that $\{\hat{u}_i\}_{i\in\mathbb{N}}$ is a Cauchy sequence in L^2. By completeness of L^2, there exists $\hat{u}$ such that $\hat{u}_i \to \hat{u}$ strongly. Moreover, it is easy to see that the limit does not depend on the approximating sequence. Let $\{\tilde{u}_i\}_{i\in\mathbb{N}}$, $\tilde{u}_i \in C_c^\infty$ such that $\tilde{u}_i \to u$ in $L^2(\mathbb{R}^n)$. Then, there exists $\hat{\tilde{u}}$ such that $\hat{\tilde{u}}_i \to \hat{\tilde{u}}$ in $L^2(\mathbb{R}^n)$. Suppose now that $\hat{\tilde{u}} \neq \hat{u}$: if so, there exists $\delta > 0$ such that $\|\hat{\tilde{u}} - \hat{u}\|_2 \geqslant \delta > 0$. But then, $\delta \leqslant \|\hat{\tilde{u}} - \hat{u}\|_2 \leqslant \|\hat{\tilde{u}} - \hat{\tilde{u}}_i\|_2 + \|\hat{\tilde{u}}_i - \hat{u}\|_2 \to 0$ as $i \to \infty$, thus giving a contradiction. Therefore, $\|\hat{\tilde{u}} - \hat{u}\|_2 = 0$.

Let us discuss some important properties of the Fourier transform in L^2.

Theorem 2.5. *Let $u, v \in L^2(\mathbb{R}^n)$. Then*

1. $\int_{\mathbb{R}^n} u\bar{v}dx = \int_{\mathbb{R}^n} \hat{u}\bar{\hat{v}}dx$,

2. $\widehat{(D^\alpha u)} = (ik)^\alpha \hat{u}$, *$\forall \alpha$ such that* $D^\alpha u \in L^2$,

3. *Let $u \in L^1$, and $\check{u}(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int dk e^{ik\cdot x} u(k)$. Let $\check{u}$ be also the extension to L^2. Then $\check{\hat{u}} = u$, $\forall u \in L^2$*

Remark 2.6. 1. *Property* 1 *implies that the standard inner product in L^2 is invariant under the Fourier transform.*

2. *Property* 2 *is the main reason why the Fourier transform is important for the PDEs: after Fourier transform, differential operators become multiplication operators.*

3. *$\check{u}$ is called the inverse Fourier transform of u.*

Proof. 1. Apply the Plancherel theorem to $\|u + \alpha v\|_2$, with $\alpha = 1, i$.

2. Suppose that $u \in C_c^\infty$. Then

$$\begin{aligned}\widehat{(D^\alpha u)}(k) =& \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ik\cdot x} D^\alpha u(x) dx = \frac{(-1)^\alpha}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \left(D^\alpha e^{-ik\cdot x}\right) u(x) dx = \\ =& \frac{(ik)^\alpha}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ik\cdot x} u(x) dx \equiv (ik)^\alpha \hat{u}(k)\end{aligned} \tag{2.28}$$

The general statement follows by approximating u with C_c^∞ functions.

3. The statement follows from this proposition:

Proposition 8.

$$\int \check{\hat{u}} v dx = \int uv dx, \qquad v \in L^2, u \in L^2. \tag{2.29}$$

Proof. This proposition can be proved using the density of $L^1 \cap L^2$ in L^2. Let $u, v \in L^1 \cap L^2$. Then

$$\int_{\mathbb{R}^n} \check{u} v dx = \int_{\mathbb{R}^n} u \check{v} dx, \tag{2.30}$$

and then we extend the identity to L^2 by approximation. In particular,

$$\int_{\mathbb{R}^n} \check{\hat{u}} v dx = \int_{\mathbb{R}^n} \hat{u} \check{v} dx = \int_{\mathbb{R}^n} \hat{u} \bar{\bar{\check{v}}} = \int_{\mathbb{R}^n} uv, \tag{2.31}$$

by the uniqueness of the inner product in L^2. □

□

2.4 Applications

Example 2.4.1. Consider the equation $-\Delta u + u = f$, $f \in L^2(\mathbb{R}^n)$. We look for solutions in $L^2(\mathbb{R}^n)$, $-\Delta u \in L^2(\mathbb{R}^n)$. We have

$$(1 + |k|^2)\hat{u}(k) = \hat{f}(k) \Rightarrow \hat{u}(k) = \frac{\hat{f}(k)}{1 + |k|^2}, \tag{2.32}$$

therefore

$$u(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int dk e^{ikx} \frac{\hat{f}(k)}{1+|k|^2}. \tag{2.33}$$

Recall that

$$\widehat{v_1 * v_2} = (2\pi)^{\frac{n}{2}} \hat{v}_1(k)\hat{v}_2(k) \Rightarrow v_1 * v_2 = (2\pi)^{\frac{n}{2}} \left(\hat{v}_1(k)\hat{v}_2(k)\right)^{\check{}}. \tag{2.34}$$

Therefore, we write:

$$\frac{1}{(2\pi)^{\frac{n}{2}}} \int dk\, e^{ikx} \frac{\hat{f}(k)}{1+|k|^2} = \frac{1}{(2\pi)^{\frac{n}{2}}} \left(f * B\right)(x)\,. \tag{2.35}$$

with B the inverse Fourier transform of $1/(1+|k|^2)$. This is however formal: the function $1/(1+|k|^2)$ is not in $L^2(\mathbb{R}^n)$ for $n \geqslant 2$. Let us ignore this fact for the moment. To find the function B we use the identity:

$$\frac{1}{1+|y|^2} = \int_0^\infty e^{-t(1+|y|^2)}. \tag{2.36}$$

Therefore,

$$B(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int dy\, e^{iyx} \frac{1}{1+|y|^2} = \frac{1}{(2\pi)^{\frac{n}{2}}} \int dy e^{iyx} \int_0^\infty dt\, e^{-t(1+|y|^2)}\,. \tag{2.37}$$

Interchanging the integrals:

$$B(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_0^\infty dt e^{-t} \int_{\mathbb{R}^n} e^{ikx} e^{-t|k|^2} = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_0^\infty dt e^{-t} \left(\frac{\pi}{t}\right)^{\frac{n}{2}} e^{-\frac{|x|^2}{4t}}. \tag{2.38}$$

From this expressions we see that, for $x=0$, the integral is divergent for $n \geqslant 2$. It is however finite for $x \neq 0$. The lack of regularity at $x=0$ of $B(x)$ is related with the lack of integrability at infinity of $\hat{B}(k)$. The function $B(x)$ is called Bessel potential. Finally, the solution is:

$$u(x) = \frac{(f*B)(x)}{(2\pi)^{\frac{n}{2}}} = \int_0^\infty dt \int_{\mathbb{R}^n} dy\, \frac{e^{-t-\frac{|x-y|}{4t}}}{(4\pi t)^{\frac{n}{2}}}. \tag{2.39}$$

Example 2.4.2 (Schrödinger equation)**.** Let us consider the initial value problem

$$\begin{cases} iu_t + \Delta u = 0, & \text{if } \mathbb{R}^n \times \{0,\infty\}, \\ u = g, & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases} \tag{2.40}$$

The behavior of the solution of the Schrödinger equation is different from the heat equation, due to the presence of a factor i in front of the time derivative. In order to find a solution, let us replace t with it in the solution of the heat equation, (1.255). We get:

$$u(x,t) = \frac{1}{(4\pi i t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} dy e^{i\frac{|x-y|^2}{4t}} g(y) \tag{2.41}$$

where we used $i^{\frac{1}{2}} = e^{i\frac{\pi}{4}}$. Suppose that $g \in L^1(\mathbb{R}^n)$ and that $|y|^2 g \in L^1(\mathbb{R}^n)$. Then, one can check that $iu_t + \Delta u = 0$ in $\mathbb{R}^n \times (0,\infty)$. One can also prove that $u(\cdot,t) \underset{t\to 0^+}{\to} g$ by using stationary phase methods (which we will not discuss).

Suppose that $g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. We rewrite the solution of the Shrödinger equation as

$$u(x,t) = \frac{e^{i\frac{|x|^2}{4t}}}{i^{\frac{n}{2}}} \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} dy e^{-i\frac{x\cdot y}{2t}} \tilde{g}(y), \tag{2.42}$$

with $\tilde{g}(y) = \frac{1}{(2t)^{\frac{n}{2}}} \exp\left(i\frac{|y|^2}{4t}\right) g(y)$, so that

$$u(x,t) = \frac{e^{i\frac{|x|^2}{4t}}}{i^{\frac{n}{2}}} \hat{\tilde{g}}(x/2t). \tag{2.43}$$

That is, if $g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ then $\hat{\tilde{g}}(\cdot/2t) \in L^2(\mathbb{R}^n)$. By Plancherel's theorem, $u(\cdot,t) \in L^2(\mathbb{R}^n)$. Moreover, again by Plancherel:

$$\|u(\cdot,t)\|_2 = \|\hat{\tilde{g}}(\cdot/2t)\|_2 = \|\tilde{g}(\cdot/2t)\|_2 = \left(\int dy |g(y/2t)|^2 \frac{1}{(2t)^n}\right)^{\frac{1}{2}} = \|g\|_2. \tag{2.44}$$

Therefore, the map $g \to u(\cdot,t)$ preserves the L^2 norm of the initial datum. Moreover, as we did for the Fourier transform, we can extend the solution $u(x,t)$ of the Schrödinger equation for (2.42) for $g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ to functions $g \in L^2(\mathbb{R}^n)$.

The solution of the Schrödinger equation has the form

$$u(x,t) = \frac{1}{(4\pi i t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} dy e^{i\frac{|x-y|^2}{4t}} g(y), \equiv (\psi(\cdot,t) * g)(x), \tag{2.45}$$

with $\psi(x,t) = \frac{e^{i\frac{|x^2|}{4t}}}{(4\pi i t)^{\frac{n}{2}}}$ the fundamental solution of the Schrödinger equation. This function plays the same role as $\Phi(x,t)$ for the heat equation. Notice that, due to the i factor at the exponent in the definition of ψ, the convolution $(\psi(\cdot,t) * g)(x)$ makes sense for $t < 0$ as well, in contrast to the heat equation.

Thus, $u(x,t) = (\psi(\cdot,t) * g)(x)$ solves the Schrödinger equation (2.40) for all times in $\mathbb{R}$. In particular, the Schrödinger equation is reversible in time: if $u(x,t)$ is a solution of Eq. (2.40), then $u(x,-t)$ is also a solution of Eq. (2.40).

Example 2.4.3 (The wave equation)**.** Let us now use the Fourier transform to solve the wave equation:

$$\begin{cases} u_{tt} - \Delta u = 0, & \text{in } \mathbb{R}^n \times (0,\infty), \\ u = g, u_t = h, & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases} \tag{2.46}$$

Taking the Fourier transform we get:

$$\begin{cases} \hat{u}_{tt} + |k|^2 \hat{u} = 0, & \text{in } \mathbb{R}^n \times (0,\infty), \\ \hat{u} = \hat{g}, \hat{u}_t = \hat{h} & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases} \tag{2.47}$$

Thus, after taking the Fourier transform, the initial PDE is reduced to an ODE. The solution is:

$$\hat{u}(k) = \hat{g}(k)\cos(t|k|) + \frac{\hat{h}(k)}{|k|}\sin(t|k|). \tag{2.48}$$

One easily checks:

$$\begin{aligned}\hat{u}_t(k) &= -\hat{g}(k)|k|\sin(t|k|) + \hat{h}(k)\cos(t|k|),\\ \hat{u}_{tt}(k) &= -\hat{g}(k)|k|^2\cos(t|k|) - \hat{h}(k)|k|\sin(t|k|) \equiv -|k|^2\hat{u}(k).\end{aligned} \tag{2.49}$$

Also,

$$\lim_{t\to 0}\hat{u}(k) = \hat{g}(k)\ , \qquad \lim_{t\to 0}\hat{u}_t = \hat{h}(k). \tag{2.50}$$

Therefore, the solution of the wave equation can be written as, taking the inverse Fourier transform:

$$u(x) = \left(\hat{g}\cos(t|k|) + \frac{\hat{h}}{|k|}\sin(t|k|)\right)^{\check{}}. \tag{2.51}$$

As a check, suppose, for simplicity, that $\hat{h} = 0$ and $n - 1$. Then

$$\begin{aligned} u(x) = (\hat{g}\cos(t|k|))^{\check{}} &= \frac{1}{(2\pi)^{\frac{1}{2}}}\int dke^{ikx}\hat{g}(k)\cos(t|k|)\\ &= \frac{1}{(2\pi)^{\frac{1}{2}}}\int dk\hat{g}(k)\cos(t|k|)\frac{1}{2}\left(e^{ikx+|t|k} + e^{ikx-|t|k}\right), \end{aligned} \tag{2.52}$$

which immediately gives d'Alembert formula.

Recall the definition of energy of the solution of the wave equation

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^n}(u_t(x)^2 + |Du(x)|^2)dx, \qquad t \geqslant 0. \tag{2.53}$$

First of all, notice that if $Dg, h \in L^2$ then the energy is finite and constant in time:

$$E(t) = E(0) = \frac{1}{2}\int_{\mathbb{R}^n}\left(h^2 + |Dg|^2\right)dx < \infty. \tag{2.54}$$

We will now prove the following proposition that tells us that the energy splits equally into kinetic and potential parts.

Proposition 9 (Equipartition of energy)**.**

$$\lim_{t\to\infty}\int_{\mathbb{R}^n}|Du(x,t)|^2dx = \lim_{t\to\infty}\int_{\mathbb{R}^n}u_t(x,t)^2dx = E(0). \tag{2.55}$$

Proof. We have

$$\begin{aligned}\int_{\mathbb{R}^n}|Du|^2dx = \int_{\mathbb{R}^n}&|y^2||\hat{u}|^2dy =\\ &= \int_{\mathbb{R}^n}dy(|y|^2|\hat{g}|^2\cos^2(t|y|) + |\hat{h}|^2\sin^2(t|y|)dy) +\\ &+ \int_{\mathbb{R}^n}dy|y|\cos(t|y|)\sin(t|y|)(\hat{h}\bar{\hat{g}} + \hat{g}\bar{\hat{h}}).\end{aligned} \tag{2.56}$$

Let $f \in C_c^\infty(\mathbb{R}^n)$. We have:

$$\begin{aligned}\int_{\mathbb{R}^n} dy \cos(t|y|)\sin(t|y|)f(y) &= \frac{1}{2}\int_{\mathbb{R}^n} dy \sin(2t|y|)f(y) = \frac{1}{2}\int_0^\infty dr \sin(2tr)\int_{\partial B(0,r)} dS f = \\ &= \frac{1}{4t}\int_0^\infty \cos(2tr)\frac{d}{dr}\int_{\partial B(0,r)} f dS dr\end{aligned} \tag{2.57}$$

where the last integral defined a compactly supported function, due to the fact that f is compactly supported. Therefore,

$$\left|\int_{\mathbb{R}^n} dy \cos(t|y|)\sin(t|y|)f(y)\right| \leqslant \frac{C}{t} \xrightarrow[t\to\infty]{} 0. \tag{2.58}$$

Now, if $Dg, h \in L^2$, then $|y|\left(\hat{h}\bar{\hat{g}} + \hat{g}\bar{\hat{h}}\right) \in L^1$. In fact:

$$\||y|\left(\hat{h}\bar{\hat{g}} + \hat{g}\bar{\hat{h}}\right)\|_1 \leqslant \||y|\hat{h}\bar{\hat{g}}\|_1 + \||y|\hat{g}\bar{\hat{h}}\|_1 \leqslant 2\||y|\hat{g}\|_2\|\hat{h}\|_2 = 2\|Dg\|_2\|h\|_2, \tag{2.59}$$

where for the last inequality we used Cauchy- Schwarz, and for the last equality we used Plancherel's theorem. Therefore, $|y|\left(\hat{h}\bar{\hat{g}} + \hat{g}\bar{\hat{h}}\right)$ can be approximated with C_c^∞ functions: for any fixed $\epsilon > 0$ there exists $\bar{i} > 0$ such that for $i \geqslant \bar{i}$, and for all $t \geqslant 0$:

$$\Big|\int_{\mathbb{R}^n} \cos(t|y|)\sin(t|y|)|y|(\hat{h}(y)\bar{\hat{g}}(y) + \hat{g}(y)\bar{\hat{h}}(y))\Big| \leqslant \Big|\int_{\mathbb{R}^n} \cos(t|y|)\sin(t|y|)f_i(y)\Big| + \epsilon\ . \tag{2.60}$$

and

$$\Big|\int_{\mathbb{R}^n} \cos(t|y|)\sin(t|y|)f_i(y)\Big| \leqslant \frac{C_i}{t}. \tag{2.61}$$

Therefore, taking t large enough,

$$\Big|\int_{\mathbb{R}^n} \cos(t|y|)\sin(t|y|)f_i(y)\Big| \leqslant 2\epsilon\ . \tag{2.62}$$

By the arbitrariness of ϵ, we conclude that:

$$\lim_{t\to\infty}\int_{\mathbb{R}^n} \cos(t|y|)\sin(t|y|)|y|(\hat{h}(y)\bar{\hat{g}}(y) + \hat{g}(y)\bar{\hat{h}}(y)) = 0\ . \tag{2.63}$$

Consider now the first two terms in the right hand side of (2.56), and recall that

$$\cos^2(t|y|) = \frac{1}{2}(\cos(2t|y|) + 1), \qquad \sin^2(t|y|) = 1 - \frac{1}{2}(\cos(2t|y|) + 1)\ . \tag{2.64}$$

Therefore, proceeding as before, we get:

$$\begin{aligned}\lim_{t\to\infty}\int_{\mathbb{R}^n} |y|^2|\hat{g}|^2\cos^2(t|y|)dy &= \frac{1}{2}\int_{\mathbb{R}^n} |y|^2|\hat{g}|^2 dy, \\ \lim_{t\to\infty}\int_{\mathbb{R}^n} |\hat{h}|^2\sin^2(t|y|)dy &= \frac{1}{2}\int_{\mathbb{R}^n} dy|\hat{h}|^2.\end{aligned} \tag{2.65}$$

In conclusion:

$$\lim_{t\to\infty}\|Du(\cdot,t)\|_2^2 = \frac{1}{2}\int_{\mathbb{R}^n} dy\left(|y|^2|\hat{g}(y)|^2 + |\hat{h}(y)|^2\right) = \frac{1}{2}\left(\|Dg\|_2^2 + \|h\|_2^2\right) = E(0). \tag{2.66}$$

$\square$

Chapter 3

Quasi-linear partial differential equations

3.1 Quasi-linear partial differential equations of first order

Let $U \subset \mathbb{R}^n$ open. Quasi-linear partial differential equation of first order are PDEs of the form:

$$F(Du, u, x) = 0, \qquad x \in U \subset \mathbb{R}^n \tag{3.1}$$

with F linear in Du. Notice that, however, F might be nonlinear in u. Equivalently, we consider PDEs of the form:

$$b(x, u(x)) \cdot Du(x) = c(u(x), x). \tag{3.2}$$

3.1.1 Homogeneous case with constant coefficients

Suppose that b is constant, *i.e.* a fixed vector in $\mathbb{R}^n$ and that $c = 0$. We have:

$$b \cdot Du = 0. \tag{3.3}$$

That is, u is constant along the direction b,

$$(b \cdot Du)(x) = \frac{d}{dt} u(x + tb)\Big|_{t=0}. \tag{3.4}$$

By changing variables, the PDE is equivalent to:

$$\frac{\partial}{\partial x_1} u(x) = 0 \Rightarrow u(x_1, x_2, \cdots, x_n) = u(0, x_2, \cdots, x_n). \tag{3.5}$$

Let us now consider the boundary value problem:

$$\begin{cases} b \cdot Du = 0, & \text{in } \mathbb{R}^n, \\ u = g, & \text{on } \Gamma_\nu \end{cases} \tag{3.6}$$

with $b \in \mathbb{R}^n$, $\nu \in \mathbb{R}^n$, $\Gamma_\nu = \{x \in \mathbb{R}^n \mid x \cdot \nu = 0\}$.

Theorem 3.1. *Let $b, \nu \in \mathbb{R}^n \backslash \{0\}$, $b \cdot \nu \neq 0$, $g \in C^1(\Gamma)$. Then, (3.6) admits a unique solution $u \in C^1(\mathbb{R}^n)$.*

Remark 3.1. *The constraint $b \cdot \nu \neq 0$ is important. To see why, suppose that $n = 2, b = e_1, \nu = e_2$. Then*

$$(b \cdot Du)(x) = \partial_{x_1} u(x_1, x_2) = 0 \Rightarrow u(x_1, x_2) = u(0, x_2) \qquad \forall x \in \mathbb{R}^n.$$

This is impossible for g non-constant, thus there are no solutions. Instead, for g=constant, if u is a solution, then $u(x) + f(x_2)$ with $f(0) = 0$ is a solution; therefore, there are infinitely many solutions.

Proof.

$$u(x) = u(x_0 + bt) = u(x_0) = g(x_0). \tag{3.7}$$

Let us call

$$t = \frac{x_n}{\cos\theta} = \frac{\langle x, \nu \rangle}{\cos\theta} = \frac{\langle x, \nu \rangle}{\langle b, \nu \rangle}. \tag{3.8}$$

Therefore,

$$u(x) = g\left(x - \frac{\langle x, \nu \rangle}{\langle b, \nu \rangle} b\right). \tag{3.9}$$

□

Remark 3.2. *The curve $x_0 + tb$, where the solution is constant, is an example of* characteristic curve, *that we will discuss later in more detail.*

Suppose now that $c \neq 0$ and b=constant. We shall consider

$$b \cdot Du(x) = c(x, u(x)). \tag{3.10}$$

Equivalently, the partial differential equation can be written as

$$\partial_{x_1} u(x) = c(x, u(x)). \tag{3.11}$$

This is an ordinary differential equation for the x_1 variable, parametrized by $x_2, x_3, \cdots x_n$. We are interested in finding a C^1 solution of the equation. Local existence and uniqueness follows from standard ODEs arguments.

Theorem 3.2. *Let $U \subset \mathbb{R}^n$ open, $b, \nu \in \mathbb{R}^n \backslash \{0\}$, $b \cdot \nu \neq 0$, $c \in C^1(\mathbb{R} \times U; \mathbb{R})$. Let*

$$\Gamma := \{x \in \mathbb{R}^n \big| x \cdot \nu = 0\}, \qquad g \in C^1(\Gamma). \tag{3.12}$$

For each $x_* \in \Gamma \cap U$ there exists $\omega \subset U$ open such that $x_* \in \omega$ and

$$\begin{cases} b \cdot Du(x) = c(u(x), x), & \text{in } \mathbb{R}^n, \\ u = g, & \text{on } \Gamma, \end{cases} \tag{3.13}$$

has a unique solution $u \in C^1(\omega)$.

This theorem follows from the following result.

Theorem 3.3. *Let $I \in \mathbb{R}$ open, $f \in C^1(I \times \mathbb{R}^n \times R^m; \mathbb{R}^n)$, $g \in C^1(\mathbb{R}^m; \mathbb{R}^n)$. Let $s_x \in I$, $z_* \in \mathbb{R}^m$. Then, $\exists \omega \subset I \times \mathbb{R}^m$ open, $I \ni (s_*, z_*)$ and $\varphi \in C^1(U; \mathbb{R}^n)$ such that*

$$\begin{cases} \partial_s \varphi(s,z) = f(s, \varphi(s,z), z), \\ \varphi(s_*, z) = g(z). \end{cases} \tag{3.14}$$

Moreover, let $\varphi_i(s,z) = \partial_{z_i} \varphi(s,z)$. Then

$$\begin{cases} \partial_s \varphi_i(s,z) = D_y f(s, \varphi(s,z), z) \varphi_i(s,z) + D_z f(s, \varphi(s,z), z) e_i, \\ \varphi_i(s_*, z) = \partial_{z_i} g(z). \end{cases} \tag{3.15}$$

In order to prove this theorem, we will use the following two well known results in ODEs.

Theorem 3.4 (Picard-Lindelhöf)**.** *Let $I = [a,b] \subset \mathbb{R}$ be a nonempty, compact interval. Let $f \in C^0(I \times \mathbb{R}^n; \mathbb{R}^n)$ be Lipschitz in the second variable, i.e.*

$$|f(x,y) - f(x,y')| \leqslant M|y - y'|.$$

Then, for all $y_0 \in \mathbb{R}^n$ the Cauchy problem

$$\begin{cases} y'(x) = f(x, y(x)), \\ y(x_0) = y_0, \qquad (x_0 \in I), \end{cases} \tag{3.16}$$

has a unique solution $\varphi \in C^1(I; \mathbb{R}^n)$.

Theorem 3.5 (Gronwall inequality)**.** *Let $I = [x_0, x_1] \subset \mathbb{R}$, $f \in C^0(I \times \mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^n)$ such that*

$$|f(x,y,z) - f(x,\bar{y},\bar{z})| \leqslant L|y - \bar{y}| + M|z - \bar{z}| \tag{3.17}$$

for all x, y, z. Consider the Cauchy problems

$$\begin{cases} \varphi'(x) = f(x, \varphi(x), z_0), \\ \varphi(x_0) = y_0, \end{cases} \quad \text{and} \quad \begin{cases} \bar{\varphi}'(x) = f(x, \bar{\varphi}(x), \bar{z}_0), \\ \bar{\varphi}(x_0) = \bar{y}_0, \end{cases} \tag{3.18}$$

Then,

$$|\varphi(x) - \bar{\varphi}(x)| \leqslant [|y_0 - \bar{y}_0| + M|z_0 - \bar{z}_0|(x_1 - x_0)] e^{L(x - x_0)}. \tag{3.19}$$

Now we have all the ingredients to prove Theorem (3.3).

Proof of Theorem (3.3). For simplicity, we shall assume that I is bounded and that $f \in C^1(\bar{I} \times \mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^m)$ is Lipschitz continuous. Also, let us assume that $s_* = 0$. The Picard-Lindelhöf theorem (3.4) implies that $\forall z \in \mathbb{R}^m$ there exists $\varphi(\cdot, z) \in \mathbb{C}^1(\bar{I}; \mathbb{R}^n)$ such that (3.14) is satisfied. Moreover, thanks to Gronwall's theorem (3.5) φ is Lipschitz in z. Using again Picard-Lindelhöf theorem we infer that (3.15) has a unique solution φ_i. We are left with proving that φ is C^1 in z, and we shall prove this by showing that

$$\partial_{z_i} \varphi = \varphi_i. \tag{3.20}$$

To prove this, consider

$$\varphi_h(s,z) := \frac{\varphi(s,z+he_i)-\varphi(s,z)}{h}. \tag{3.21}$$

Using the fact that φ is Lipschitz, we get $\|\varphi_h\|_\infty \leqslant M$. Moreover

$$\begin{aligned} h\varphi_h(s,z) &= \varphi(s,z+he_i) - \varphi(s,z) = \\ &= g(z+he_i) - g(z) + \int_0^s dt[f(t,\varphi(t,z+he_i),z+he_i) - f(t,\varphi(t,z),z)]. \end{aligned} \tag{3.22}$$

Let us rewrite the integrand as

$$\begin{aligned} &f(t,\varphi(t,z+he_i),z+he_i) - f(t,\varphi(t,z),z) = f(t,\varphi+h\varphi_h,z+he_i) - f(t,\varphi(t,z),z) = \\ =&f(t,\varphi+h\varphi_h,z+he_i) - f(t,\varphi(t,z),z+he_i) + f(t,\varphi(t,z),z+he_i) - f(t,\varphi(t,z),z) = \\ =&h\overline{D_yf}(t)\varphi_h + h\overline{D_zf}(t)\cdot e_i, \end{aligned} \tag{3.23}$$

where we used

$$\varphi_h \equiv \varphi_h(t,z), \qquad \varphi \equiv \varphi(t,z), \tag{3.24}$$

and

$$\overline{D_yf}(t) = \int_0^1 d\lambda D_yf(t,\varphi+\lambda h\varphi_h, z+he_i), \qquad e_i\cdot\overline{D_zf}(t) = \int_0^1 e_i\cdot D_zf(t,\varphi,z+\lambda he_i)d\lambda. \tag{3.25}$$

Therefore,

$$h\varphi_h(s,z) = \varphi(s,z+he_i) - \varphi(s,z) = g(z+he_i) - g(z) + h\int_0^s [\overline{D_yf}(t)\varphi_h + \overline{D_zf}(t)e_i]dt. \tag{3.26}$$

The boundedness of φ_h implies

$$\lim_{h\to0}\overline{D_yf}(t) = D_yf(t,\varphi,z), \qquad \lim_{h\to0}\overline{D_zf}(t) = D_zf(t,\varphi,z). \tag{3.27}$$

Let us now rewrite the equation for φ_i appearing in the statement of the theorem in integral form:

$$\varphi_i(s,z) = Dg(z)\cdot e_i + \int_0^s [Dyf(t,\varphi,z)\varphi_i(t,z) + e_i\cdot D_zf(t,\varphi,z)]dt, \tag{3.28}$$

and the goal is to show that $\|\varphi_h - \varphi_i\|_\infty \xrightarrow[h\to0]{} 0$. We have

$$\begin{aligned} \varphi_i - \varphi =& \frac{g(z+he_i)-g(z)}{h} - e_i\cdot Dg(z) + \int_0^s dt\left(\overline{D_yf}\varphi_h - D_yf\varphi_i + \overline{D_zf}\cdot e_i - D_zf\cdot e_i\right) = \\ =& \frac{g(z+he_i)-g(z)}{h} - e_i\cdot Dg(z) + \\ &+ \int_0^s dt\left[\left(\overline{D_yf}\varphi_h - D_yf\varphi_i\right)\varphi_h + D_yf(\varphi_h-\varphi_i) + \left(\overline{D_zf} - D_zf\right)\cdot e_i\right]. \end{aligned} \tag{3.29}$$

Therefore:

$$\begin{aligned} \sup_{I\times B_1(z_*)}\|\varphi_h - \varphi\|_\infty \leqslant& \left\|\frac{g(z+he_i)-g(z)}{h} - e_i\cdot Dg(z)\right\|_\infty + \\ &+|I|\|\overline{D_yf} - D_yf\|_\infty M + |I|\|D_yf\|_\infty\|\varphi_h-\varphi_i\|_\infty + |I|\|\overline{D_zf} - D_zf\|_\infty. \end{aligned} \tag{3.30}$$

Finally, taking $|I|$ small enough so that

$$|I|\|D_y f\|_\infty \leqslant \frac{1}{2} \tag{3.31}$$

we conclude

$$\lim_{h\to 0} \|\varphi_h - \varphi\|_\infty = 0 \tag{3.32}$$

□

3.1.2 Non-homogeneous case with non-constant coefficients

Let us start by supposing that $b(u(x), x) \equiv b(x)$. Consider the equation:

$$b(x) \cdot Du(x) = c(u(x), x) \; . \tag{3.33}$$

To solve it, we proceed as follows. We look for a curve $\gamma \in C^1(I; U)$ such that

$$\gamma'(t) = b(\gamma(t)). \tag{3.34}$$

If so, using that

$$\frac{d}{dt} u(\gamma(t)) = \gamma'(t) \cdot Du(\gamma(t)) \tag{3.35}$$

the PDE reduces to

$$\begin{cases} \frac{d}{dt} u(\gamma(t)) = c(u(\gamma(t)), \gamma(t)), \\ \gamma'(t) = b(\gamma(t)), \end{cases} \tag{3.36}$$

i.e. we reduced the PDE to a system of ODEs.

Example 3.1.1. Let $U = \mathbb{R}^2$, $b(x) = (-x_2, x_1)$. For all $r > 0$, the curve $\gamma(t)$ we are looking for is $\gamma(t) = (r\cos t, r\sin t)$. Consider the equation

$$\frac{d}{dt} u(\gamma(t)) = c(u(\gamma(t)), \gamma(t)). \tag{3.37}$$

Depending on c, it may or may not have a global solution.

1. Case $c = 0$. The solution u is constant along γ. The curve γ is called a characteristic curve of the equation. We have:
$$\frac{d}{dt} u(\gamma(t)) = 0 \Rightarrow u(x) \equiv u(|x|). \tag{3.38}$$

2. Case $c = 1$. In this case, the equation has no global solution: $u(\gamma(t))$ is strictly increasing and $\gamma(t + 2\pi) = \gamma(t)$. However the solution still exists locally, *i.e.* for small t.

Remark 3.3. *This example shows that this method of solving PDEs depends strongly on the geometry of the characteristic curve $\gamma(t)$. If $\gamma(t)$ intersects itself at later times, there might be an obstruction in defining the solution at the intersection of the characteristics.*

Now, suppose that b depends on the solution u as well. We consider the equation:

$$b(u(x), x) \cdot Du(x) = c(u(x), x) \ . \tag{3.39}$$

We would like to proceed as we did for the previous case. This time, however, to know who $\gamma'(s)$ is, we need to know both $\gamma(s)$ and $u(\gamma(s))$. Thus, we shall look at curves on a larger space. Let $\Gamma(s) = (u(\gamma(s)), \gamma(s))$, for $\gamma(s)$ to be determined. We are interested in solutions of the ODE:

$$\begin{cases} \frac{d}{ds}\Gamma(s) = \frac{d}{ds}\begin{pmatrix}\Gamma_0(s)\\ \gamma(s)\end{pmatrix} = \begin{pmatrix}c(\Gamma(s))\\ b(\Gamma(s))\end{pmatrix}, \\ \Gamma(0) = (u_*, x_*) \qquad \text{with } u_* = u(x_*). \end{cases} \tag{3.40}$$

Lemma 3.1. *Let $U \subset \mathbb{R}^n$ open, $b \in C^1(\mathbb{R} \times U; \mathbb{R}^n)$, $c \in C^1(\mathbb{R} \times U)$, $x_* \in U$, $u_* \in \mathbb{R}$. Let $\Gamma \in C^1(-\delta, \delta)$, $\delta > 0$ be a solution of (3.40), and let $u \in C^1(U)$ be a solution of*

$$b(u(x), x) \cdot Du(x) = c(u(x), x), \qquad u(x_*) = u_x. \tag{3.41}$$

Then, $u(\gamma(s)) = \Gamma_0(s)$ for all $s \in (-\delta, \delta)$.

Proof. Let $\varphi(s) := u(\gamma(s)) - \Gamma_0(s)$. Then, $\varphi(0) = 0$ and

$$\begin{aligned} \varphi' &= Du \circ \gamma \cdot \gamma' - c \circ \Gamma = \\ &= Du \circ \gamma \cdot b(u \circ \gamma, \gamma) + Du \circ \gamma \cdot (\underbrace{b(\Gamma_0, \gamma)}_{\gamma'} - b(u \circ \gamma, \gamma)) - c(\Gamma_0, \gamma) = \\ &= c(u \circ \gamma, \gamma) - c(\Gamma_0, \gamma) + Du \circ \gamma \cdot (b(\Gamma_0, \gamma) - b(u \circ \gamma, \gamma)) \ . \end{aligned} \tag{3.42}$$

Therefore,

$$\begin{cases} \frac{d}{ds}\varphi(s) = G(s, \varphi(s)), \\ \varphi(0) = 0, \end{cases} \tag{3.43}$$

with

$$G(\cdot, y) = c(\Gamma_0 + y, y) - c(\Gamma_0, \gamma) + Du \circ \gamma \cdot (b(\Gamma_0, \gamma) - b(\Gamma_0 + y, \gamma)), \tag{3.44}$$

and

$$\Gamma_0(\cdot) + \varphi(\cdot) = u(\gamma(\cdot)) \ . \tag{3.45}$$

Notice that $y = 0$ is a zero of $G(\cdot, y)$. Therefore, by Picard-Lindelhöf, Theorem 3.4, there exists $\delta > 0$ such that the ODE (3.43) admits a unique solution for $s \in (-\delta, \delta)$. Since $\varphi(s) = 0$ is a solution of Eq. (3.43), we proved that $u(\gamma(s)) = \Gamma_0(s)$ for all $s \in (-\delta, \delta)$. □

We are now in the position to prove the following theorem.

Theorem 3.6. *Let $U \subset \mathbb{R}^n$, $J \subset \mathbb{R}$ open, $b \in C^1(J \times U; \mathbb{R}^n)$, $c \in C^1(J \times U)$, $x_* \in U$, $g(x_*) \in J$, $b_n(g(x_*, x_*)) \neq 0$. Then, there exists $\omega \ni x_*$ open such that the initial value problem*

$$\begin{cases} b(u(x), x) \cdot Du(x) = c(u(x), x), & \text{in } \omega, \\ u = g & \text{on } \omega \cap \{x_n = x_n^*\} \end{cases} \tag{3.46}$$

has a unique solution $u \in C^1(\omega)$.

Remark 3.4. *The set $\{x \in \mathbb{R}^n \mid x_n = x_n^*\}$ is what we called Γ_ν before (the normal ν is now e_n). The condition $b_n(g(x_*), x_*) \neq 0$ is the analog of $b \cdot \nu \neq 0$.*

Proof. For $z \in \mathbb{R}^{n-1}$, $|z|$ small enough, let $\Gamma(\cdot) \equiv \Gamma(z, \cdot) \in C^1(-\delta, \delta)$ be the unique solution of:

$$\frac{d}{ds}\Gamma(s) = \begin{pmatrix} c(\Gamma(s)) \\ b(\Gamma(s)) \end{pmatrix}, \qquad \Gamma(0) = \begin{pmatrix} g(x_* + (z,0)) \\ x_* + (z,0), \end{pmatrix} \tag{3.47}$$

Let $\gamma(z,s) := \gamma(\Gamma(s))$. Since $\gamma(z,0) = x_* + z$ and $(d/ds)\gamma(z,0) = b(g(x_* + z), x_* + z)$, we get

$$\det D\gamma(0,0) \neq 0 , \tag{3.48}$$

thus $(z,s) \mapsto \gamma(z,s)$ is invertible in a neighborhood of $(0,0)$. In fact:

$$D\gamma(0,0) = \begin{pmatrix} 1 & 0 & & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \\ b_1 & b_2 & \cdots & b_n \end{pmatrix} \tag{3.49}$$

with $b \equiv b(g(x_*), x_*)$ and $b_n \neq 0$ by assumption. The eigenvalues of the matrix can be read off from the diagonal, and are all nonzero. Let now $\psi = \gamma^{-1}$, and let us define $u := \Gamma_0 \circ \psi$. Then, since $\Gamma_0 \in C^1$ and $\psi \in C^1$ (locally) we have $u \in C^1(\omega)$, for some ω open such that $\omega \ni x_*$. By construction, $u \circ \gamma = \Gamma_0$.

We are left with checking that u solves the original PDE. Let $s = y_n$ and $z = (y_1, \cdots, y_{n-1})$. We compute

$$\begin{aligned} \partial_s u(\gamma(z,s)) &= Du(\gamma(z,s)) \cdot \partial_s \gamma(z,s) = Du(\gamma(y)) \cdot b(\gamma(y), \Gamma_0(y)) \equiv \\ &\equiv Du(\gamma(y)) \cdot b(\gamma(y), u(\gamma(y))), \end{aligned} \tag{3.50}$$

and

$$\partial_s \Gamma_0(y,s) = c(\gamma(y), \Gamma_0(y)) = c(\gamma(y), u(\gamma(y))) \; ; \tag{3.51}$$

therefore,

$$Du(\gamma(y)) \cdot b(u(\gamma(y)), \gamma(y)) = c(u(\gamma(y)), \gamma(y)) \; . \tag{3.52}$$

□

3.1.3 Burgers equation

Let us consider the following PDE:

$$\begin{cases} \partial_t u + \partial_x \frac{u^2}{2} = 0, & \mathbb{R} \times (0, \infty) , \\ u(x,0) = g(x) & x \in \mathbb{R} . \end{cases} \tag{3.53}$$

The unknown is $u \equiv u(x,t)$, $(x,t) \in \mathbb{R} \times (0,\infty)$. In order to rewrite the equation as in Eq. (3.39) and apply the theory developed in the previous section, we set $b(u(y), y) = (u(y), 1)$, $c = 0$. Eq. (3.53) is equivalent to:

$$b(u(y), y) \cdot Du(y) = 0 \; . \tag{3.54}$$

Let us solve the equation using the method of characteristics. The associated ODE is:

$$\frac{d}{ds}\Gamma(s) = \begin{pmatrix} 0 \\ b(\Gamma(s)) \end{pmatrix} = \begin{pmatrix} 0 \\ \Gamma_0(s) \\ 1 \end{pmatrix} , \qquad \Gamma(0) = \begin{pmatrix} .g(x) \\ x \\ 0 \end{pmatrix} \tag{3.55}$$

The solution is:

$$\Gamma_0(s) = g(x) , \qquad \gamma(s) = \begin{pmatrix} x + g(x)s \\ s \end{pmatrix} \equiv \begin{pmatrix} x(s) \\ t(s) \end{pmatrix} . \tag{3.56}$$

Let us go back to the initial value problem (3.53), for a specific choice of the boundary condition g:

$$g(x) = \begin{cases} 1 & \text{for } x \leqslant 0, \\ 1 - x & \text{for } x \in (0,1), \\ 0 & \text{for} x \geqslant 1 \end{cases} \tag{3.57}$$

From the proof of the previous theorem, we know that the solution $u(x,t)$ can be found, locally, as $\Gamma_0(\psi(x,t))$. We have

$$u(\gamma(s)) = \Gamma_0(s) = g(x) , \qquad \gamma(s) \equiv \gamma(x,s) = (x + g(x)s, s). \tag{3.58}$$

To find the function $u(x,t)$, we consider different values of x.

- **Case** $x \leqslant 0$**.** By (3.57) we know that $g(x) = 1$, so

$$u(x+s, s) = g(x) . \tag{3.59}$$

  Setting $y := x + s$, $u(y,s) = g(y-s)$ for $y - s \leqslant 0$. That is:

$$u(y,s) = 1 \qquad \text{for } y \leqslant s. \tag{3.60}$$

- **Case** $x \in (0,1)$**.** By (3.57), $g(x) = 1 - x$. Thus,

$$u(x + (1-x)s, s) = g(x) = 1 - x . \tag{3.61}$$

  Let $y := x + (1-x)s = x(1-s) + s$. We have $x = \frac{y-s}{1-s}$, and hence:

$$u(y,s) = 1 - \frac{y-s}{1-s} = \frac{1-y}{1-s} \qquad \text{if } 0 < \frac{1-y}{1-s} < 1. \tag{3.62}$$

- **Case** $x \geqslant 1$**.** By (3.57), $g(x) = 0$, therefore

$$u(x,s) = 0 \qquad \text{for } x \geqslant 1. \tag{3.63}$$

All in all

$$u(x,t) = \begin{cases} 1 & \text{if } x \leqslant t \\ \frac{x-t}{1-t} & \text{if } 0 \leqslant \frac{1-x}{1-t} \leqslant 1 \\ 0 & \text{if } x \geqslant 1. \end{cases} \tag{3.64}$$

From Eq. (3.62), we immediately see that the solution is not defined for $t \geqslant 1$. The reason is that the map $(z,s) \mapsto \gamma(z,s)$ is not invertible for $s \geqslant 1$; instead, the characteristic curves intersect at $s = 1$. Therefore, the method of characteristics only provides a solution for small times, $t < 1$.

3.1.4 Weak solutions for conservation laws

The previous example motivates the introduction of a more general notion of solution. More generally, we shall consider PDEs of the form:

$$\begin{cases} u_t + \partial_x F(u) = 0 & (x,t) \in \mathbb{R} \times (0,\infty) \\ u(x,0) = g(x) & x \in \mathbb{R} \,. \end{cases} \tag{3.65}$$

This equation is called a *conservation law.*

Definition 30. *Let $F \in C^1(\mathbb{R})$, $g \in L^\infty(\mathbb{R})$, $u \in L^\infty(\mathbb{R} \times (0,\infty))$. The function u is a weak solution of the equation (3.65) if, for all $v \in C_c^\infty(\mathbb{R} \times [0,\infty))$,*

$$\int_{\mathbb{R}\times[0,\infty)} (u\partial_t v + F \circ u \partial_x v)\, dxdt + \int_{\mathbb{R}} gv(\cdot,0)dx = 0 \,. \tag{3.66}$$

The function v is called a *test function.* In order to understand this definition, suppose that u solves (3.65) is the classical (or strong) sense:

$$\partial_t u + \partial_x F(u) = 0. \tag{3.67}$$

Then, trivially:

$$\int_{\mathbb{R}\times[0,\infty)} dxdt\, (\partial_t u + F(u)_x)v = 0 \,. \tag{3.68}$$

Let us now integrate by parts:

$$0 = \int_{\mathbb{R}\times[0,\infty)} dxdt\; (-u\partial_t v - F(u)\partial_x v) + \int_{\mathbb{R}} dx\, uv\big|_{t=0}^{t=\infty} \,. \tag{3.69}$$

Therefore, using the compact support of v:

$$0 = \int_{\mathbb{R}\times[0,\infty)} dxdt\, (u\partial_t v + F(u)\partial_x v)(x,t) + \int_{\mathbb{R}} dxg(x)v(x,0) \,. \tag{3.70}$$

The advantage of this notion of solution is that Eq. (3.66) makes sense even if u is not in C^1.

Remark 3.5. *It is easy to see that a weak solution of class C^1 is a strong solution.*

Theorem 3.7. *Let $u \in L^\infty(\mathbb{R} \times [0,\infty))$ be a weak solution of (3.65), and let $\omega \subset \mathbb{R} \times \overline{(0,\infty)}$ open, $U \subset \mathbb{R} \times \overline{(0,\infty)}$ open with C^1 boundary. Suppose there exist two functions $u_1 \in C^1(\overline{\omega \cap U})$ and $u_2 \in C^1(\overline{\omega \backslash U})$ such that $u = u_1$ on $\omega \cap U$ and $u = u_2$ on $\omega \backslash \bar{U}$. Then*

$$\begin{pmatrix} F(u_1) - F(u_2) \\ u_1 - u_2 \end{pmatrix} \cdot \nu = 0 \qquad \text{on } \partial U \cap \omega \,, \tag{3.71}$$

with ν the normal of ∂U.

Let $\nu = (\nu_1, \nu_2)$. Then, Eq. (3.71) reads:

$$(u_1 - u_2)\nu_2 + (F(u_1) - F(u_2))\nu_1 = 0 \,. \tag{3.72}$$

Now, let us represent parametrically the boundary $(\partial U) \cap \omega$ as: $\{(x,t) \mid x = s(t)\}$ for some smooth $s(\cdot) : [0,\infty) \to \mathbb{R}$. Then, the tangent to the boundary is $(\dot{s}(t), 1)$, while the normal is $\frac{1}{\sqrt{\dot{s}(t)^2+1}}(1, -\dot{s}(t))$. Therefore, Eq. (3.72) becomes

$$F(u_1) - F(u_2) = \dot{s}(u_1 - u_2). \tag{3.73}$$

$F(u_1) - F(u_2)$ is the jump of $F(u)$ across the curve, while $u_1 - u_2$ is the jump of u across the curve. The ratio between the jumps os equal to the *the speed* of the curve, $\dot{s}$. This is called the *Rankine-Hugoniot condition.* Let us now prove Theorem 3.7.

Proof. By definition of weak solution,

$$0 = \int_0^\infty \int_{-\infty}^\infty dxdt\, uv_t + F(u)v_x \tag{3.74}$$

with v a test function with support in ω. Then

$$\int_{\omega\cap U} dxdt\, uv_t + F(u)v_x + \int_{\omega\setminus U} dxdt\, uv_t + F(u)v_x = 0. \tag{3.75}$$

By Gauss-Green theorem, we get:

$$\begin{aligned}\int_{\omega\cap U} uv_t + F(u)v_x = \int_{\omega\cap U} (-\partial_t uv - \partial_x F(u)v) + \int_{\partial\omega\cap U} (F(u), u)\cdot \nu v = \\ \equiv \int_{\partial\omega\cap U} (F(u), u)\cdot \nu v,\end{aligned} \tag{3.76}$$

where we used that $\partial_t u + \partial_x F(u) = 0$ in $\omega \cap U$. Similarly,

$$\int_{\omega\setminus U} (uh_t + F(u)h_x) = -\int_{\partial\omega\cap U} (F(u), u)\cdot \nu h\ . \tag{3.77}$$

All in all, we get

$$\int_{\partial\omega\cap U} v\nu\cdot (F(u_1) - F(u_2), u_1 - u_2) = 0\ . \tag{3.78}$$

By the arbitrariness of v, $\nu\cdot(F(u_1) - F(u_2), u_1 - u_2) = 0$ on $\partial\omega \cap U$, which concludes the proof. □

Thus, the Rankine-Hugoniot condition gives a necessary condition for a function u to be a weak solution. The next theorem shows that this condition is also sufficient.

Theorem 3.8. *Let $F \in C^1(\mathbb{R})$, $g \in L^\infty(\mathbb{R})$, $u \in L^\infty(\mathbb{R}\times(0,\infty))$. Suppose that there exists countably many open sets U_i in $\mathbb{R}^2$ with C^1 boundary and functions $u_i \in C^1(\bar{U}_i)$, such that $\mathbb{R}\times[0,\infty) \subset \bigcup_i U_i$, $u = u_i$ on U_i, $u_i = g$ on $\overline{U}_i \cap (\mathbb{R}\times\{0\})$. Then, u is a weak solution of Eq. (3.65) if and only if*

$$\partial_t u + \partial_x F(u) = 0 \qquad \text{on all } U_i \tag{3.79}$$

and the Rankine-Hugoniot condition holds true on all ∂U_i.

Example 3.1.2. Consider again the Burgers equation

$$\begin{cases} \partial_t u + \partial_x \frac{u^2}{2} = 0, & \text{in } \mathbb{R} \times (0, \infty), \\ u = g, & \text{on } \mathbb{R} \times \{t = 0\} \end{cases} \tag{3.80}$$

with

$$g(x) = \begin{cases} 1, & x \leqslant 0, \\ 1 - x, & x \in [0, 1], \\ 0, & x > 1. \end{cases} \tag{3.81}$$

We would like to extend for all times the solution found in Section 3.1.3. Recall:

$$u(x, t) = \begin{cases} 1, & x \leqslant t, 0 \leqslant t < 1, \\ \frac{1-x}{1-t}, & t \leqslant x \leqslant 1, 0 \leqslant t < 1, \\ 0, & x \geqslant 1, 0 \leqslant t < 1. \end{cases} \tag{3.82}$$

We define a continuation of the above solution, using the RH condition:

$$u(x, t) = \begin{cases} 1, & x < f(t), \\ 0, & f(t) < x, \end{cases} \qquad t \geqslant 1. \tag{3.83}$$

where f is the characteristic function

$$f(t) = \frac{1 + t}{2}. \tag{3.84}$$

This function naturally extends Eq. (3.82) to all times, and satisfies the RH condition. It defines a *weak solution* of the Burgers equation.

As the next example will show, weak solutions are, in general, not unique.

Example 3.1.3. Consider again the Burgers equation (3.53), this time with initial condition

$$g(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0, \end{cases} \tag{3.85}$$

Using the method of characteristics, we get:

$$u(x, t) = 1, \quad x - t > 0, \tag{3.86}$$

and

$$u(x, t) = 0, \qquad x \leqslant 0. \tag{3.87}$$

However, the method of characteristics cannot be used to determine the solution for $t \geqslant x \geqslant 0$. Let us use the RH condition to continue the solution in this region. We define

$$u(x, t) = \begin{cases} 0, & x < \frac{t}{2}, \\ 1, & x > \frac{t}{2}. \end{cases} \tag{3.88}$$

One immediately checks that:

$$\frac{1}{2}\underbrace{(u_1 - u_1)}_{=1} = \underbrace{F(u_1) - F(u_2)}_{=\frac{1}{2}} . \tag{3.89}$$

Therefore, u defines a weak solution of the Burgers equation. However, we could have proceeded differently. We define:

$$\tilde{u}(x,t) = \begin{cases} 1, & x > t, \\ \frac{x}{t}, & 0 < x < t, \\ 0, & x < 0, \end{cases} \tag{3.90}$$

that continuously interpolates between 0 and 1 in the wedge $t \geqslant x \geqslant 0$. In this last case, the solution is continuous at $x = t$ and $x = 0$ (while the derivative is not). It is easy to check that the solution satisfies the Burgers equation, away from $x = t$ and $x = 0$. Also, it trivially satisfies the RH condition on $x = t$ and $x = 0$. Thus $\tilde{u}$ defines a weak solution of the Burgers equation. It takes the name of *rarefaction wave*.

Presumably, uniqueness holds true in a subclass of weak solutions, that satisfy some extra conditions motivated by Physics. Remember the ODE satisfied by the characteristics:

$$\frac{d}{ds}\begin{pmatrix} x(s) \\ t(s) \end{pmatrix} = \begin{pmatrix} \Gamma_0(s) \\ 1 \end{pmatrix}, \qquad \frac{d}{ds}\Gamma_0(s) = 0 \Rightarrow \dot{x}(s) = \Gamma_0(s) = u(x(s), s) . \tag{3.91}$$

The quantity $\dot{x}(s)$ takes the name of *speed* of the solution. In the case of the Burgers equation, it is equal to the solution itself: "taller" waves move faster than "shorter" waves. In particular, for $u = 0$ the speed is zero. Recall now the first solution, Eq. (3.88). There, at the disontinuity $x = t/2$ the wave $u = 0$ and the wave $u = 1$ move with the same speed, which is unphysical. Instead, for the second solution Eq. (3.90), the wave $u = 0$ does not move, as it should. In general, we only allow to discontinuities if the leftmost part of the wave at the discontinuity is moving faster than the rightmost part of the wave. Consider a general conservation law $\partial_t u + F(u)_x = 0$. Calling u_- and u_+ the left and right part of the wave at the discontinuity, we require that:

$$F'(u_-) > F'(u_+) . \tag{3.92}$$

This is equivalent to the requirement that discontinuities might only form in the future. This condition takes the name of *entropy condition*. A discontinuity curve satisfying the entropy condition and the RH condition is called a *shock curve*. The corresponding weak solution is called an *entropy solution*. It is easy to see that in Example (3.1.3) the entropy condition is violated:

$$u_- = 0, \quad u_+ = 1, \quad F'(u_-) = 0, \quad F'(u_+) = 1 \Rightarrow F'(u_-) < F'(u_+) . \tag{3.93}$$

Thus, the physical solution of the Burgers equation is $\tilde{u}$.

Finally, let us quickly mention some important results about conservation laws. It turns out that if F is smooth and uniformly convex, $F'' \geqslant \theta > 0$. In this case, F' is increasing and in particular

$$F'(u_-) > F'(u_+) \Leftrightarrow u_- > u_+ . \tag{3.94}$$

For this class of PDEs it is possible to prove that there is at most one entropy solution, up to a set of measure zero (that is, if there are two solutions u and $\tilde{u}$, then $u = \tilde{u}$ a.e.).

Chapter 4

Second order elliptic PDEs

In this section we shall introduce a general class of second order partial differential equations. More precisely, we shall consider boundary value problems of the form:

$$\begin{cases} Lu = f, & \text{in } U, \\ u = 0, & \text{on } \partial U, \end{cases} \tag{4.1}$$

with $U \subset \mathbb{R}^n$ and f given. The unknown is $u \equiv u(x)$. Lu denotes the action of a linear map L on u, which we assume of the form:

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x) u_{x_i x_j} + \sum_{j=1}^{n} b^i(x) u_{x_i} + c(x) u(x), \tag{4.2}$$

for given functions $a^{ij}(x)$, $b^i(x)$, $c(x)$. We shall be mostly interested in the situation in which the PDE is *elliptic*.

Definition 31 (Elliptic PDE)**.** *The PDE operator L is (uniformly) elliptic if there exists $\theta > 0$ such that*

$$\sum_{i,j} a^{ij}(x) \xi_i \xi_j \geqslant \theta |\xi|^2, \tag{4.3}$$

almost everywhere in x and $\forall \xi \in \mathbb{R}^n$.

Remark 4.1. *If $a^{ij} = \delta_{ij}$, Eq. (4.2) reduces to the action of the Laplace operator on u, that is Eq. (4.1) reduces to Laplace's problem.*

In general, it might be hopeless to find explicit formulas for solutions of the boundary value problem (4.1). As for the conservation laws discussed in the previous chapter, in order to develop a theory of solutions of Eq. (4.1) we will need to weaken the notion of solution. The "right" spaces where to look for weak solutions of the boundary value problem (4.1) are provided by *Sobolev spaces*, discussed in the next section.

4.1 Sobolev spaces

In this section we give a brief introduction to Sobolev spaces. The reader is encouraged to check [Lieb-Loss] and [Evans] for more details. We start with the following definition.

Definition 32. *We define the space $L^p_{loc}(U)$ as:*

$$L^p_{loc}(U) := \{u : U \to \mathbb{R} \mid u \in L^p(V) \text{ for all } V \subset\subset U\} . \tag{4.4}$$

Definition 33 (Weak derivative)**.** *Suppose $u, v \in L^1_{loc}(U)$ and let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multi-index. We say that $v \in L^1_{loc}(U)$ is the $\alpha-$th weak derivative of u, written $D^\alpha u = v$, if:*

$$\int_U u D^\alpha \phi \, dx = (-1)^\alpha \int_U v\phi \, dx, \tag{4.5}$$

for all test functions $\phi \in C^\infty_c(U)$.

Remark 4.2. *If $u \in C^\alpha(U)$ then $v \in C(U)$ is the usual α-th derivative of u. (integrate by parts, and use the arbitrariness of ϕ).*

Lemma 4.1. *The weak α-th partial derivative of u, if it exists, it is uniquely defined up to a set of measure zero.*

Proof. Let $u \in L^1_{\mathrm{loc}}(U)$. Suppose that $v, \tilde{v} \in L^1_{\mathrm{loc}}(U)$ satisfy

$$\int_U u D^\alpha \phi = (-1)^\alpha \int_U v\phi = (-1)^\alpha \int \tilde{v} \,, \qquad \forall \phi \in C^\infty_c(U) . \tag{4.6}$$

Then:

$$\int (v - \tilde{v})\phi = 0 \qquad \forall \phi \in C^\infty_c(U) . \tag{4.7}$$

Thus, $v = \tilde{v}$ up to a set of measure zero. ☐

Example 4.1.1. Let $n = 1$, $U = (0, 2)$ and

$$u(x) = \begin{cases} x, & 0 < x \leqslant 1, \\ 1, & 1 \leqslant x < 2. \end{cases} \tag{4.8}$$

The function is not differentiable in the classical sense. Nevertheless, it admits a weak derivative. Let

$$v(x) = \begin{cases} 1, & 0 < x \leqslant 1, \\ 0, & 1 < x < 2. \end{cases} \tag{4.9}$$

We claim that $u' = v$ in the weak sense. Let $\phi \in C^\infty_c(U)$. We want to show that

$$\int_0^2 u\phi' dx = -\int_0^2 v\phi dx . \tag{4.10}$$

By explicitly computing the integral:

$$\int_0^2 u\phi' dx = \int_0^1 x\phi' dx + \int_1^2 \phi' dx = -\int_0^1 \phi dx + \phi(1) + \phi(2) - \phi(1) = -\int_0^1 \phi dx = -\int_0^2 v\phi dx \,, \tag{4.11}$$

where we used $\phi(2) = 0$ since $\phi \in C^\infty_c((0, 2))$. This proves the claim.

Example 4.1.2. Let $n = 1$, $U = (0, 2)$ and

$$u(x) = \begin{cases} x, & 0 < x \leqslant 1, \\ 2, & 1 < x < 2. \end{cases} \tag{4.12}$$

We claim that u' does not exist in the weak sense: there exists no function $v \in L^1_{\text{loc}}(U)$ such that

$$\int_0^2 u\phi' dx = -\int_0^2 v\phi dx \qquad \forall \phi \in C_c^\infty \; . \tag{4.13}$$

Suppose that (4.13) is true for some v and for all ϕ. Then

$$\begin{aligned} -\int_0^2 v\phi dx &= \int_0^2 u\phi' dx = \int_0^1 x\phi' dx + 2\int_1^2 \phi' dx \\ &= -\int_0^1 \phi dx + \phi(1) + 2\phi(2) - 2\phi(1) = -\int_0^1 \phi dx - \phi(1), \end{aligned} \tag{4.14}$$

where we only used $\phi(2) = 0$. Choose now a sequence of functions $\{\phi_m\}_{m=1}^\infty$, $\phi_m \in C_c^\infty(U)$ such that

$$0 \leqslant \phi_m \leqslant 1, \qquad \phi_m(1) = 1, \qquad \phi_m(x) \to 0 \qquad \forall x \neq 1. \tag{4.15}$$

Then

$$-\int_0^2 v\phi_m dx = -\int_0^1 \phi_m dx - \phi_m(1) \Rightarrow 1 = \lim_{m\to\infty} \phi_m(1) = \lim_{m\to\infty} \left(\int_0^2 v\phi_m - \int_0^1 \phi_m \right) = 0 \tag{4.16}$$

since $\phi_m(x) \to 0$ for all $x \neq 1$, which is a contradiction.

Now we are ready to define Sobolev spaces.

Definition 34 (Sobolev space)**.** *The Sobolev space $W^{k,p}(U)$ consists of all functions $u \in L^1_{loc}(U)$ such that for all α, $|\alpha| < k$, $D^\alpha u$ exists in the weak sense and $D^\alpha u \in L^p(U)$:*

$$W^{k,p}(U) := \{u : U \to \mathbb{R} \mid D^\alpha u \text{ exists in the weak sense}, \quad D^\alpha u \in L^p(U), \quad \text{for all } |\alpha| \leqslant k\} \tag{4.17}$$

Remark 4.3. 1. *If $p = 2$, we write $H^k(U) = W^{k,2}(U)$. The notation is motivated by the fact that H^k is a Hilbert space.*

2. *As for L^p spaces, functions in $W^{k,p}$ are identified if they coincide almost everywhere.*

Next, we list some straightforward properties of weak derivatives and of Sobolev spaces.

Proposition 10. *Let $u, v \in W^{k,p}(U)$, $|\alpha| \leqslant k$. Then*

1. *$D^\alpha u \in W^{k-|\alpha|,p}(U)$, $D^\beta(D^\alpha u) = D^\alpha(D^\beta u) = D^{\alpha+\beta} u$, $\forall \alpha, \beta$ such that $|\alpha| + |\beta| \leqslant k$.*
2. *$\forall \lambda, \mu \in \mathbb{R}$, $\lambda u + \mu v \in W^{k,p}(U)$ and $D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v$, for all $|\alpha| \leqslant k$.*
3. *Let $V \subset U$, V open. Then $u \in W^{k,p}(V)$.*
4. *Let $\zeta \in C_c^\infty(U)$. Then $\zeta u \in W^{k,p}(U)$ and $D^\alpha(\zeta u) = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} D^\beta \zeta D^{\alpha-\beta} u$ where $\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$ (Leibniz rule).*

Proof. Left to the reader. □

As we shall see, Sobolev spaces are Banach spaces. In order to prove this, we need to introduce a norm in $W^{k,p}$.

Definition 35 (Sobolev norm)**.** *Let $u \in W^{k,p}(U)$. We define the Sobolev norm $\|\cdot\|_{W^{k,p}(U)}$ as*

$$\|u\|_{W^{k,p}(U)} = \begin{cases} \left(\sum_{|\alpha|\leqslant k} \int_U |D^\alpha u|^p dx\right)^{1/p}, & 1 \leqslant p < \infty, \\ \sum_{|\alpha|\leqslant k} \text{ess } sup_U |D^\alpha u|, & p = \infty. \end{cases} \tag{4.18}$$

Remark 4.4. *1. One can check that $\|\cdot\|_{W^{k,p}(U)}$ satisfies all the properties a norm must have:*

- $\|\lambda u\|_{W^{k,p}(U)} = |\lambda|\|u\|_{W^{k,p}(U)}$ *for all* $\lambda \in \mathbb{R}$,
- $\|u\|_{W^{k,p}(U)} = 0 \iff u = 0$ *in* $W^{k,p}(U)$,
- $\|u+v\|_{W^{k,p}(U)} \leqslant \|u\|_{W^{K,p}(U)} + \|v\|_{W^{k,p}(U)}$

2. The Sobolev norm is expressed in terms of the L^p norms of the weak derivatives $D^\alpha u$. As for L^p spaces, if we did not identify functions differing on sets of measure zero, $\|\cdot\|_{W^{k,p}(U)}$ would only define a seminorm.

Example 4.1.3. Let $U = B(0,1)$ the unit ball in $\mathbb{R}^n$. Define $u(x) := |x|^{-a}$, for $x \in U, x \neq 0$. Let us find the values of $a > 0, k, p$ for which u belongs to $W^{1,p}(U)$. We compute

$$u_{x_i}(x) = -\frac{ax_i}{|x|^{a+2}}, \qquad x \neq 0\ . \tag{4.19}$$

This implies:

$$|Du(x)| = \frac{a}{|x|^{a+1}}, \qquad x \neq 0. \tag{4.20}$$

Thus, u is only differentiable in $B(0,1)\backslash\{0\}$. Let us check that the function u admits a weak derivative in $B(0,1)$. Let $\phi \in C_c^\infty(U)$, $\epsilon > 0$. Then

$$\int_{U\backslash B(0,\epsilon)} u\phi_{x_i} dx = -\int_{U\backslash B(0,\epsilon)} u_{x_i}\phi + \int_{\partial B(0,\epsilon)} u\phi\nu^i, \tag{4.21}$$

where ν is the inward normal of $\partial B(0,\epsilon)$. Suppose that $a+1 < n$. Then, $|Du(x)| \in L^1(U)$, which implies:

$$\Big|\int_{\partial B(0,\epsilon)} u\phi\nu^i ds\Big| \leqslant \|\phi\|_\infty \int_{\partial B(0,\epsilon)} \epsilon^{-a} ds \leqslant C\epsilon^{n-1-a} \xrightarrow[\epsilon\to 0]{} 0. \tag{4.22}$$

Therefore, since both u and Du are in $L^1(U)$ we have:

$$\int_U u\phi_{x_i} dx = \lim_{\epsilon\to 0}\int_{U\backslash B(0,\epsilon)} u\phi_{x_i} dx = -\int_U u_{x_i}\phi dx, \qquad \forall\phi \in C_c^\infty(U),\ 0 \leqslant a < n-1. \tag{4.23}$$

This proves that u_{x_i} is the weak derivative of u in U. Moreover, $|Du(x)| = \frac{a}{|x|^{a+1}} \in L^p(U) \Leftrightarrow (a+1)p < n$. Therefore

$$u \in W^{1,p}(U) \Leftrightarrow a < \frac{n-p}{p}\ . \tag{4.24}$$

The next example shows that functions in Sobolev spaces can be rather badly behaved.

Example 4.1.4. Let $\{r_k\}_{k\in\mathbb{N}}$ a countable, dense subset of $U = B(0,1)$. Define

$$u(x) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{1}{|x - r_k|^a}, \qquad x \in U. \tag{4.25}$$

By density of $\{r_k\}_{k\in\mathbb{N}}$ in $B(0,1)$, the function u is unbounded on all open subsets of U. Nevertheless, we claim that if $0 < a \leqslant \frac{n-p}{p}$ then $u \in W^{1,p}(U)$.

As we have seen in the previous example, $\frac{1}{|\cdot - r_k|^a} \in W^{1,p}(U)$. Also, a finite linear combination of $\frac{1}{|\cdot - r_k|}$ still belongs to $W^{k,p}$. Let us now check that the function u belongs to $W^{k,p}$. We have:

$$\|u\|_{W^{1,p}(U)} \leqslant \sum_{k=1}^{\infty} \frac{1}{2^k} \left\| \frac{1}{|\cdot - r_k|^a} \right\|_{W^{1,p}(U)} \leqslant C, \qquad 0 \leqslant a \leqslant \frac{n-p}{p} \; . \tag{4.26}$$

Let us now introduce a notion of convergence in $W^{k,p}(U)$.

Definition 36. *1. Let $\{u_m\}_{m=1}^{\infty}$, $u \in W^{k,p}(U)$. We say that u_m converges to u in $W^{k,p}(U)$,*

$$u_m \to u \; , \qquad \text{in } W^{k,p}(U) \; , \tag{4.27}$$

if

$$\lim_{m\to\infty} \|u_m - u\|_{W^{k,p}(U)} = 0 \; . \tag{4.28}$$

2. *We write $u_m \to u$ in $W^{k,p}_{loc}(U)$ to mean $u_m \to u$ in $W^{k,p}(V)$ for any $V \subset\subset U$.*

3. *We denote by $W_0^{k,p}(U)$ the closure of $C_c^\infty(U)$ in $W^{k,p}(U)$. That is, $u \in W_0^{k,p}(U)$ if and only if there exist $\{u_m\}_{m\in\mathbb{N}}$, $u_m \in C_c^\infty$, such that $u_m \to u$ in $W^{k,p}(U)$.*

Finally, the next theorem shows that $W^{k,p}(U)$ is Banach spaces.

Theorem 4.1. *$\forall k \in \mathbb{N}$, $1 \leqslant p \leqslant \infty$, the space $W^{k,p}(U)$ is a Banach space.*

Proof. We have to check that $W^{k,p}(U)$ is a complete, normed linear space. We already proved linearity and we introduced a norm. Let us check completeness. Let $\{u_m\}_{m=1}^{\infty}$ be a Cauchy sequence in $W^{k,p}(U)$. Then, $\forall |\alpha| \leqslant k$, $\{D^\alpha u_m\}_{m=1}^{\infty}$ is a Cauchy sequence in $L^p(U)$. By completeness of $L^p(U)$, $\exists u_\alpha \in L^p(U)$ such that $D^\alpha u_m \to u_\alpha$ in $L^p(U)$ for any $|\alpha| \leqslant k$. In particular, $u_m \to u_{(0,0,\cdots,0)} =: u$. We claim that $D^\alpha u = u_\alpha$ for any $|\alpha| \leqslant k$. This would prove that $D^\alpha u \in L^p(U)$, and hence that $u \in W^{k,p}(U)$, which is what we want to show.

Let $\phi \in C_c^\infty(U)$. Then

$$\int_U u D^\alpha \phi = \lim_{m\to\infty} \int_U u_m D^\alpha \phi. \tag{4.29}$$

This identity immediately follows from Hölder inequality:

$$\|(u - u_m) D^\alpha \phi\|_1 \leqslant \|u - u_m\|_{L^p} \|D^\alpha \phi\|_{L^q}, \qquad \frac{1}{p} + \frac{1}{q} = 1 \; . \tag{4.30}$$

Therefore, recalling the definition of weak derivative:

$$\int_U uD^\alpha\phi = \lim_{m\to\infty}(-1)^\alpha \int_U D^\alpha u_m \phi dx \equiv (-1)^{|\alpha|}\int_U u_\alpha \phi dx, \tag{4.31}$$

where in the last step we used that $D^\alpha u_m \to u_\alpha$ in $L^p(U)$. This proves that $D^\alpha u = u_\alpha$, as claimed. □

It is natural to ask whether Sobolev functions can be approximated with smooth functions, for which weak derivatives reduce to standard derivatives. In order to show this, recall the definition of mollifier (Chapter 1, Eq. (1.105)).

We will start by proving an interior approximation theorem, that provides a smooth approximations for Sobolev functions not uniformly in the distance from the boundary of U.

Theorem 4.2 (Interior approximation by smooth functions)**.** *Let $u \in W^{k,p}(U)$ for some $1 \leqslant p < \infty$ and set $u^\epsilon = \eta_\epsilon * u$ in U_ϵ. Then:*

1. *$u^\epsilon \in C^\infty(U_\epsilon)$, $\qquad \forall \epsilon > 0$,*
2. *$u^\epsilon \to u$ in $W^{k,p}_{loc}(U)$, $\epsilon \to 0$.*

Proof. 1. Recall that

$$u^\epsilon(x) = \int dy\, \eta_\epsilon(x-y)u(y), \qquad \forall x \in U_\epsilon.$$

Therefore,

$$D^\alpha u^\epsilon(x) = \int dy\, D^\alpha_x \eta_\epsilon(x-y)u(y), \tag{4.32}$$

where integral and derivative can be interchanged, thanks to the fact that $|D^\alpha_x \eta_\epsilon(x-\cdot)|$ is bounded uniformly in x and $u \in L^1_{\text{loc}}(U)$. This proves the first part of the theorem.

2. To begin, we claim that $D^\alpha u^\epsilon = \eta_\epsilon * D^\alpha u$ in U_ϵ, $\forall|\alpha| \leqslant k$. Let us postpone for a moment the proof of this claim, and let us show how it implies that $u^\epsilon \to u$ in $W^{k,p}_{\text{loc}}(U)$ as $\epsilon \to 0$. We shall use that, for $f \in L^p_{\text{loc}}(U)$:

$$\|f^\epsilon\|_{L^p(V)} \leqslant \|f\|_{L^p(W)}, \qquad \forall V \subset\subset W \subset\subset U. \tag{4.33}$$

In fact,

$$\begin{aligned} |f^\epsilon(x)| = \Big|\int_{B(x,\epsilon)} \eta_\epsilon(x-y)f(y)dy\Big| &\leqslant \int_{B(x,\epsilon)} (\eta_\epsilon(x-y))^{1-1/p}(\eta_\epsilon(x-y))^{1/p}|f(y)|dy \\ &\leqslant \left(\int_{B(x,\epsilon)} \eta_\epsilon(x-y)dy\right)^{1-1/p}\left(\int_{B(x,\epsilon)} \eta_\epsilon(x-y)|f(y)|^p dy\right)^{1/p}, \end{aligned} \tag{4.34}$$

where the last step follows from Hölder's inequality. Thus

$$\begin{aligned} \int_V dx|f^\epsilon(x)|^p &\leqslant \int_V dx \int_{B(x,\epsilon)} dy\, \eta_\epsilon(x-y)|f(y)|^p \leqslant \\ &\leqslant \int_W dy\, |f(y)|^p \int dx\, \eta_\epsilon(x-y) = \int_W |f(y)|^p\ , \end{aligned} \tag{4.35}$$

which proves Eq. (4.33).

Now, let $f = D^\alpha u$, $f^\epsilon = \eta_\epsilon * D^\alpha u$. Let $V \subset\subset W \subset\subset U_\epsilon$, $\delta > 0$, $g \in C(W)$ such that $\|D^\alpha u - g\|_{L^p(W)} \leqslant \delta$. Then:

$$\begin{aligned}\|\eta_\epsilon * D^\alpha u - D^\alpha u\|_{L^p(V)} &\leqslant \|\eta_\epsilon * D^\alpha u - \eta_\epsilon * g\|_{L^p(V)} + \|\eta_\epsilon * g - g\|_{L^p(V)} + \|g - D^\alpha u\|_{L^p(V)} \\ &\leqslant 2\|D^\alpha u - g\|_{L^p(W)} + \|\eta_\epsilon * g - g\|_{L^p(V)} \\ &\leqslant 3\delta, \qquad \text{if } \epsilon \text{ is small enough.}\end{aligned} \tag{4.36}$$

Therefore, we just proved that

$$\eta_\epsilon * D^\alpha u \underset{\epsilon\to 0}{\longrightarrow} D^\alpha u \qquad \text{in } L^p_{\mathrm{loc}}(U). \tag{4.37}$$

Let $V \subset\subset U_\epsilon$. Eq. (4.36) implies:

$$\|u^\epsilon - u\|^p_{W^{k,p}(V)} = \sum_{|\alpha|\leqslant k} \|D^\alpha u^\epsilon - D^\alpha u\|^p_{L^p(V)} \underset{\epsilon\to 0}{\longrightarrow} 0, \tag{4.38}$$

which proves 2. To conclude, we are left with showing that

$$D^\alpha u^\epsilon = \eta_\epsilon * D^\alpha u, \qquad \text{in } U_\epsilon. \tag{4.39}$$

Let $x \in U_\epsilon$. Then

$$D^\alpha u(x) = D^\alpha \int_U \eta_\epsilon(x-y)u(y)dy = \int_U D^\alpha_x \eta_\epsilon(x-y)u(y)dy = (-1)^{|\alpha|} \int_U D^\alpha_y \eta_\epsilon(x-y)u(y)dy. \tag{4.40}$$

Now, notice that $\phi(y) := \eta_\epsilon(x-y)$ is a test function. Therefore, by definition of weak derivative

$$\int_U D^\alpha_y \eta_\epsilon(x-y)u(y)dy = (-1)^{|\alpha|} \int_U \eta_\epsilon(x-y)D^\alpha u(y)dy\ . \tag{4.41}$$

Plugging this into Eq. (4.40) we get:

$$D^\alpha u^\epsilon(x) = (-1)^{|\alpha|+|\alpha|} \int_U \eta_\epsilon(x-y)D^\alpha u(y)dy = (\eta_\epsilon * D^\alpha u)(x)\ , \tag{4.42}$$

which proves the claim, and concludes the proof of the theorem.

□

The next step is to find approximations for functions in $W^{k,p}(U)$ by C^∞ functions, that holds true in the $W^{k,p}(U)$ rather than $W^{k,p}_{\mathrm{loc}}(U)$ sense. In other words, we are looking for approximations that are uniform in the distance from the boundary of U. In fact, what we proved so far is that for all $V \subset U$, $\overline{V}$ such that $\overline{V} \subset V$, $\overline{V}$ compact, and for all $\delta > 0$ there exists $\epsilon_0 \equiv \epsilon_0(V, \delta)$ such that $\|u_\epsilon - u\|_{W^{k,p}(V)} < \delta$ for any $\epsilon < \epsilon_0$. The reason for this lack of uniformity is that, a priori, Sobolev functions might be badly behaved in proximity of their domain U.

The next theorem provides a *global approximation result*, that holds uniformly in the distance from ∂U (but *not* on ∂U).

Theorem 4.3 (Global approximation by smooth functions)**.** *Let U open and bounded, $u \in W^{k,p}(U)$, $1 \leqslant p < \infty$. Then, there exist functions $u_m \in C^\infty(U) \cap W^{k,p}(U)$ such that $u_m \to u$ in $W^{k,p}(U)$.*

Proof. Let $U_i = \{x \in U \mid \operatorname{dist}(x, \partial U) > 1/i\}, i = 1, 2, \cdots$. Then, $U = \bigcup_i U_i$. Also, let $V_i = U_{i+3} \backslash \overline{U}_{i+1}$. Notice that V_i is an open set, that $V_i \subset\subset U$, and that $V_i \cap V_{i+1} \neq \varnothing$. Moreover, we can choose V_0 such that $V_0 \subset\subset U$ and $U = \bigcup_{i=0}^{\infty} V_i$.

Let us use Theorem 4.2 to find an approximation for u in each V_i. To this end, let us introduce a smooth partition of unity, that is a family of function $\{\zeta_i\}_{i=0}^{\infty}$, such that:

$$\begin{cases} 0 \leqslant \zeta_i \leqslant 1, & \zeta_i \in C_c^\infty(V_i), \\ \sum_{i=0}^{\infty} \zeta_i = 1, & \text{in } U. \end{cases} \tag{4.43}$$

Then, $\zeta_i u \in W^{k,p}(U)$ and $\operatorname{supp}(\zeta_i u) \subset V_i$. Let $u^i := \eta_{\epsilon_i} * (\zeta_i u)$. Notice that the support of u^i is slightly bigger than the support of $\zeta_i u$: for ϵ_i small enough, $\operatorname{supp} u^i \subset W_i$, with $W_i = U_i \backslash \overline{U}_{i+4}$. Thanks to Theorem 4.2, for all $\delta > 0$ and for ϵ_i small enough,

$$\|u^i - \rho_i u\|_{W^{k,p}(V_i)} \equiv \|u^i - \rho_i u\|_{W^{k,p}(V_i)} \leqslant \frac{\delta}{2^{i+1}}. \tag{4.44}$$

Now, write $v := \sum_{i=0}^{\infty} u^i$. Notice that, $\forall V \subset\subset U$, there exists only finitely many i's such that $u^i \neq 0$ in V. Therefore, $D^\alpha v = \sum_{i=0}^{\infty} D^\alpha u^i$, and hence $v \in C^\infty(U)$. Thus, for all $V \subset\subset U$:

$$\|v - u\|_{W^{k,p}(V)} \leqslant \sum_i \|u^i - \zeta_i u\|_{W^{k,p}(V)} \leqslant \delta \sum_i \frac{1}{2^{i+1}} = \delta. \tag{4.45}$$

To conclude, notice that δ is independent of V. Therefore, taking the supremum over V, we finally get $\|u - v\|_{W^{k,p}(U)} \leqslant \delta$, which concludes the proof. □

The previous result does not say anything about the regularity of the approximating function on the boundary: we do not know whether $v \in C^\infty(\overline{U})$. In order to prove regularity on the boundary, one needs extra information on the regularity properties of ∂U.

Theorem 4.4 (Global approximation up to the boundary)**.** *Let U be bounded and with ∂U of class C^1. Suppose $u \in W^{k,p}(U)$, $1 \leqslant p < \infty$. Then, there exists $\{u_m\}_{m=1}^{\infty}$, $u_m \in C^\infty(\overline{U})$ such that*

$$u_m \to u, \ \text{in } W^{k,p}(U). \tag{4.46}$$

Proof. See [Evans]. □

Thus, the set of C^∞ functions in $W^{k,p}(U)$ is *dense* in $W^{k,p}(U)$. The result can also be extended to unbounded subsets of $\mathbb{R}^n$. The intuitive reason is that any function $u \in W^{k,p}(U)$, with U possibly unbounded, can be approximated by ζu, with $\zeta \in C^\infty$ defined as:

$$\zeta = \begin{cases} 1 & \text{in } X, \\ 0 & \text{in } Z \backslash X \end{cases} \tag{4.47}$$

with $X \subset Z \subset U$ and X bounded. More precisely,

$$\|u - \zeta u\|_{W^{k,p}(U)} \leqslant \|u\|_{W^{k,p}(U \backslash X)}, \tag{4.48}$$

which can be made arbitrarily small by choosing X large enough. Thus, one can repeat the approximation arguments discussed above for ζu; we omit the details.

One first application of these results is about the characterization of Sobolev spaces in terms of the Fourier transform. In the following, we shall focus on the Sobolev spaces $H^k(U) := W^{k,2}(U)$. The notation is motivated by the fact that these spaces are Hilbert spaces.

Theorem 4.5. *Let $k \in \mathbb{N}$. Then the following is true:*

1. *$u \in L^2(\mathbb{R}^n)$ belongs to $H^k(\mathbb{R}^n)$ iff $(1+|y|)^k \hat{u} \in L^2(\mathbb{R}^n)$.*
2. *There exists $C > 0$ such that*

$$\frac{1}{C}\|u\|_{H^k(\mathbb{R}^n)} \leqslant \|(1+|y|)^k \hat{u}\|_{L^2(\mathbb{R}^n)} \leqslant C\|u\|_{H^k(\mathbb{R}^n)}, \qquad \forall u \in H^k(\mathbb{R}^n). \tag{4.49}$$

Proof. Let $u \in H^k(\mathbb{R}^n)$. Then, $\forall \alpha$ such that $|\alpha| \leqslant k$, $D^\alpha u \in L^2(\mathbb{R}^n)$. Suppose first that $u \in C_c^k$. Then, by the properties of the Fourier transform:

$$\widehat{D^\alpha u} = (iy)^\alpha \hat{u}. \tag{4.50}$$

For general $u \in H^k(\mathbb{R}^n)$, $\widehat{D^\alpha u}$ exists by the extension of the Fourier transform to $L^2(\mathbb{R}^n)$. The identity (4.50) holds by approximation. Suppose that (4.50) is false, that is $\exists U \subset \mathbb{R}^n$ bounded such that

$$\|\widehat{D^\alpha u} - (iy)^\alpha \hat{u}\|_{L^2(U)} > \delta. \tag{4.51}$$

Let $u_m \in C_c^\infty(\mathbb{R}^n) \cap W^{k,2}(\mathbb{R}^n)$ be an approximating sequence for u. Then

$$\begin{aligned} \|\widehat{D^\alpha u} - (iy)^\alpha \hat{u}\|_{L^2(U)} &= \|D^\alpha \widehat{(u + u_m} - u_m) - (iy)^\alpha(\hat{u} + \hat{u}_m - \hat{u}_m)\|_{L^2(U)} = \\ &= \|D^\alpha \widehat{(u - u_m)} - (iy)^\alpha(\hat{u} - \hat{u}_m)\|_{L^2(U)} \\ &\leqslant \|D^\alpha(u - u_m)\|_{L^2(\mathbb{R}^n)} + C\|u - u_m\|_{L^2(\mathbb{R}^n)} \underset{m\to\infty}{\longrightarrow} 0, \end{aligned} \tag{4.52}$$

which contradicts (4.51). Therefore, $\widehat{D^\alpha u} = (iy)^\alpha \hat{u}$ and in particular $(iy)^\alpha \hat{u} \in L^2(\mathbb{R}^n)$. This implies that

$$(1 + |y|^k)\hat{u} \in L^2(\mathbb{R}^n), \tag{4.53}$$

and

$$\begin{aligned} \int dy (1+|y|^k)^2 |\hat{u}|^2 \leqslant C \int dy (1 + |y|^{2k})|\hat{u}|^2 &= C \int dy \left(1 + \left(\sum_j y_j^2\right)^k\right)|\hat{u}|^2 \\ &\leqslant \tilde{C} \int dy \left(1 + \sum_j y_j^{2k}\right)|\hat{u}|^2 = \tilde{C}\|u\|^2_{H^k(\mathbb{R}^n)}, \end{aligned} \tag{4.54}$$

where we used that $\widehat{\partial^k u} = (ik_j)^k \hat{u}$ and the Plancherel theorem. Therefore, we proved that

$$u \in H^k(\mathbb{R}^n) \Rightarrow (1 + |y|^k)\hat{u} \in L^2(\mathbb{R}^n), \tag{4.55}$$

and in particular that

$$\|(1 + |y|^k)\hat{u}\|_{L^2(\mathbb{R}^n)} \leqslant C\|y\|_{H^k(\mathbb{R}^n)}. \tag{4.56}$$

Let us now prove the converse result. Let $(1+|y|^k)\hat{u} \in L^2(\mathbb{R}^n)$, and suppose that $|\alpha| \leqslant k$. Then

$$\|(iy)^\alpha \hat{u}\|^2_{L^2(\mathbb{R}^n)} \leqslant C\|(1+|y|^k)\hat{u}\|^2_{L^2(\mathbb{R}^n)}. \tag{4.57}$$

Let $u_\alpha := ((iy)^\alpha \hat{u})\check{}$. Using the properties of the Fourier transform we have:

$$\int_{\mathbb{R}^n} (D^\alpha \phi)\bar{u}dx = \int_{\mathbb{R}^n} \widehat{(D^\alpha \phi)}\bar{\hat{u}}dy = \int_{\mathbb{R}^n} (iy)^\alpha \hat{\phi}\bar{\hat{u}}dy = (-1)^{|\alpha|}\int_{\mathbb{R}^n} \phi((iy)^\alpha \hat{u})\check{}dy. \tag{4.58}$$

Therefore, recalling the definition of weak derivative, $u_\alpha = D^\alpha u$, and in particular $D_\alpha u \in L^2(\mathbb{R}^n)$, which implies that $u \in H^k(U)$. Also, by (4.57)

$$\|D^\alpha u\|^2_{L^2(\mathbb{R}^n)} \leqslant C\|(1+|y|^k)\hat{u}\|^2_{L^2(\mathbb{R}^n)}, \tag{4.59}$$

that is:

$$\|u\|^2_{H^k(U)} \leqslant \hat{C}\|(1+|y|^k)\hat{u}\|_{L^2(\mathbb{R}^n)}. \tag{4.60}$$

□

The Fourier transform can be used to define *fractional* Sobolev spaces, as follows.

Definition 37 (Fractional Sobolev spaces)**.** *Let $0 < s < \infty$, $u \in L^2(\mathbb{R}^n)$. Then, we say that $u \in H^s(\mathbb{R}^n)$ if $(1+|y|^s)\hat{u} \in L^2(\mathbb{R}^n)$. For non integer s, we define*

$$\|u\|_{H^s(\mathbb{R}^n)} = \|(1+|y|^s)\hat{u}\|_{L^2(\mathbb{R}^n)}. \tag{4.61}$$

4.2 Existence and uniqueness for second order elliptic PDEs

4.2.1 Weak solutions

Let $U \subset \mathbb{R}^n$ open and bounded. We consider the boundary value problem:

$$\begin{cases} Lu = f, & \text{in } U, \\ u = 0, & \text{on } \partial U, \end{cases} \tag{4.62}$$

with $f : U \to \mathbb{R}$ given. Let L be a linear differential operator, having either the form

$$Lu = -\sum_{i,j=1}^n (a^{ij}(x)u_{x_i})_{x_j} + \sum_{i=1}^n b^i(x)u_{x_i} + c(x)u, \tag{4.63}$$

or

$$Lu = -\sum_{i,j=1}^n a^{ij}(x)u_{x_i x_j} + \sum_{i=1}^n b^i(x)u_{x_i} + c(x)u, \tag{4.64}$$

for given a^{ij}, b^j, c. In the following, we shall suppose that $a^{ij} = a^{ji}$. We say that $Lu = f$ is in divergence form if L is given by (4.63), and it in non-divergence form if it is given by (4.64). If $a^{ij} \in C^1(U)$, the two definitions are equivalent, up to a redefinition of $b^i(x)$. We shall consider *elliptic second order PDEs*, meaning that there exists $\theta > 0$ such that:

$$\sum_{i,j=1}^n a^{ij}(x)\xi_i\xi_j \geqslant \theta|\xi|^2, \tag{4.65}$$

almost everywhere in x, and for all $\xi \in \mathbb{R}^n$.

Remark 4.5. *This means that the symmetric matrix*

$$A(x) = (a^{ij}(x))_{1 \leqslant i \leqslant j \leqslant n}, \tag{4.66}$$

is positive definite, i.e. its smallest eigenvalue is greater or equal than $\theta > 0$*.*

In general, it might be impossible to find solutions for such boundary value problems. Thus, we shall relax the notion of solvability, and consider *weak solutions* for Eq. (4.62). At first, suppose that u is a smooth function, and that u solves Eq. (4.62), with L in divergence form. Then, we trivially have:

$$\int_U \left(\sum_{i,j} a^{ij}(x) u_{x_i} v_{x_j} + \sum_{i=1}^n b^i u_{x_i} + cuv \right) dx = \int_U f v dx, \tag{4.67}$$

for any $v \in C_c^\infty(U)$. In fact, integrating by parts,

$$\int_U (Lu - f) v dx = 0, \tag{4.68}$$

with no boundary terms thanks to the Dirichlet boundary conditions. We shall say that u solves Eq. (4.62) is the weak sense if:

$$\int (Lu - f) v dx = 0, \qquad \forall v \in C_c^\infty(U). \tag{4.69}$$

If $u \in C_c^\infty$, this reduces to the usual notion of solution. In order to make sure that the integrals make sense, we shall suppose that a^{ij}, $b^j, c \in L^\infty(U)$.

Let us define the space:

$$H_0^1(U) := \left\{ u \in H^1(U) \mid \exists \{u_m\}_{m \in \mathbb{N}}, u_m \in C_c^\infty(U) \cap H^1(U) \text{ s.t. } u_m \to u \text{ in } H^1(U) \right\} . \tag{4.70}$$

As we shall see, this is the natural space to look for a weak solution of Eq. (4.62).

Definition 38. 1. *The linear form* $B[\cdot, \cdot]$ *associated with the elliptic operator* L *in divergence form is*

$$B[u, v] := \int_U \sum_{i,j}^n a^{ij} u_{x_i} v_{x_j} + \int_U \sum_i b^i u_{x_i} v + \int_U cuv, \tag{4.71}$$

for $u, v \in H_0^1(U)$*.*

2. *We say that* $u \in H_0^1(U)$ *is a weak solution of the boundary value problem if*

$$B[u, v] = (f, v), \qquad \forall v \in H_0^1(U), \tag{4.72}$$

where

$$(f, v) = \int dx f(x) v(x). \tag{4.73}$$

Thus, the problem of proving the existence of a weak solution is equivalent to finding the solution of a certain integral equation.

Let us discuss the last definition. Suppose $u \in C^\infty(U)$ is a usual, or strong, solution of the problem. Then, Eq. (4.69) hold true for all $\forall v \in C_c^\infty(U)$. Integrating by parts, and using the Dirichlet boundary condition, we easily get

$$B[u,v] = (f,v), \qquad \forall v \in C_c^\infty(U). \tag{4.74}$$

More generally, let v be in $H_0^1(U)$ and let $v_m \in C_c^\infty(U) \cap H_0^1(U)$. Then, $v_m \to v$ in $H^1(U)$ also implies $B[u,v_m] \underset{m\to\infty}{\longrightarrow} B[u,v]$ and $(f,v_m) \underset{m\to\infty}{\longrightarrow} (f,v)$. Thus, the identity $B[u,v] = (f,v)$ is a trivial consequence of u being a solution of $Lu = f$ and of the convergence in $H^1(U)$.

The space $H_0^1(U)$ is the largest space for which $B[u,v] = (f,v)$ makes sense, for all $v \in H_0^1(U)$. We have:

$$|B[u,v]| \leqslant \int_U \sum_{i,j=1}^n |a^{ij}(x)||u_{x_i}(x)||v_{x_j}(x)| + \sum_i \int_U dx |b^i(x)||u_{x_i}(x)||v(x)| + \int_U |c(x)||u(x)||v(x)|. \tag{4.75}$$

By Cauchy-Schwarz inequality,

$$\begin{aligned}(4.75) &\leqslant \sum_{i,j=1}^n \|a^{ij}(x)\|_\infty \|u_{x_i}\|_2 \|v_{x_j}\|_2 + \sum_i \|b^i(x)\|_\infty \|u_{x_i}\|_2 \|v\|_2 + \|c\|_\infty \|u\|_2 \|v\|_2 \\ &\leqslant C\|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)} < \infty.\end{aligned} \tag{4.76}$$

4.2.2 Lax-Milgram theorem

The next theorem will provide an important tool to prove existence and uniqueness of weak solutions for elliptic second order PDEs.

Theorem 4.6 (Lax-Milgram)**.** *Let H be a Hilbert space. Let $B : H \times H \to \mathbb{R}$ be a bilinear mapping, for which there exist $\alpha, \beta > 0$ such that*

1. $|B[u,v]| \leqslant \alpha \|u\|_H \|v\|_H \qquad \forall u, v \in H.$
2. $\beta \|u\|^2 \leqslant B[u,u] \qquad \forall u \in H.$

Let $f : H \to \mathbb{R}$ be a bounded linear functional on H. Then, there exists a unique $u \in H$ such that

$$B[u,v] = \langle f, v\rangle, \qquad \forall v \in H. \tag{4.77}$$

Coming back to the weak formulation of the boundary value problem (4.62), the Hilbert space H is $H_0^1(U)$, with U bounded and open, and the bilinear functional is given by (4.71). In this case, the pairing of f with v, $\langle f, v\rangle$, is also equal to the inner product (f,v). Thus, to prove existence and uniqueness of the solution we are left with proving the assumptions of the theorem.

Remark 4.6. *Notice that in the case $B[\cdot,\cdot]$ is symmetric, the statement of the theorem is an immediate application of Riesz representation principle. In fact, one just notices that $B[u,v]$ defines an inner product on H. By the Riesz representation theorem, we know that there is an isomorphism between the inner product defined by $B[\cdot,\cdot]$ and the linear maps acting on H. In other words, for any bounded linear functional $f : H \to \mathbb{R}$ there exists a unique $u \in H$ such that*

$$B[u,v] = \langle f, v\rangle \equiv f(v). \tag{4.78}$$

The significance of the Lax-Milgram theorem is that it does not require the map B to be symmetric.

Proof. 1. Let $u \in H$. Then $v \to B[u,v]$ is a bounded linear functional on H. By Riesy representation theorem there exist a unique $w \in H$ such that

$$B[u,v] = (w,v), \qquad v \in H. \tag{4.79}$$

Therefore, this defines a mapping

$$A : H \to H, \qquad Au = w \Rightarrow B[u,v] = (Au,v). \tag{4.80}$$

2. We claim that $A : H \to H$ is a bounded linear operator. Let $\lambda_1, \lambda_2 \in \mathbb{R}$, $u_1, u_2 \in H$. Then, for all $v \in H$:

$$(A(\lambda_1 u_1 + \lambda_2 u_2), v) = B[\lambda_1 u_1 + \lambda_2 u_2, v] = \lambda_1 B[u_1, v] + \lambda_2 B[u_2, v], \tag{4.81}$$

by linearity of B. Then

$$(4.81) = \lambda_1 (Au_1, v) + \lambda_2 (Au_2, v) = (\lambda_1 Au_1 + \lambda_2 Au_2, v) \tag{4.82}$$

by linearity of the inner product. The equality is true for all $v \in H$. Therefore,

$$A(\lambda_1 u_1 + \lambda_2 u_2) = \lambda_1 Au_1 + \lambda_2 Au_2 \Rightarrow A \text{ is linear.} \tag{4.83}$$

Moreover, it is bounded:

$$||Au||^2 = (Au, Au) = B[u, Au] \leqslant \alpha \|u\| \|Au\| \Rightarrow \|Au\| \leqslant \alpha \|u\| \Rightarrow A \text{ is bounded.} \tag{4.84}$$

3. We claim that
   (a) A is a one-to-one map,
   (b) the *range* of A, $R(A) := \{v \in H \mid v = Au \text{ for some } u \in H\}$ is closed in H.

   We first prove (a). We need to show that

$$u_1 \neq u_2 \Leftrightarrow Au_1 \neq Au_2. \tag{4.85}$$

Using point 2), we know that

$$\|A(u_1 - u_2)\| \leqslant \alpha \|u_1 - u_2\| \tag{4.86}$$

that is $Au_1 \neq Au_2 \Rightarrow u_1 \neq u_2$. To prove the converse, we compute

$$\beta \|u\|^2 \leqslant B[u,u] = (Au, u) \leqslant \|Au\| \|u\| \Rightarrow \beta \|u_1 - u_2\| \leqslant \|Au_1 - Au_2\|, \tag{4.87}$$

that proves (a). Let us now prove (b). Let $\{v_j\}_{j \in \mathbb{N}}$, $v_j \in R(A)$, be a convergent sequence in H:

$$\|v_j - v\| \underset{j \to \infty}{\longrightarrow} 0. \tag{4.88}$$

We claim that $v = Au$ for some $u \in H$, that is $v \in R(A)$. Let $u_j \in H$ such that $v_j = Au_j$. By Eq. (4.87):

$$\beta \|u_j - u_{j'}\| \leqslant \|Au_j - Au_{j'}\| = \|v_j - v_{j'}\| \underset{j,j' \to \infty}{\longrightarrow} 0, \tag{4.89}$$

which proves that u_j is a Cauchy series. Being H a Banach space, $u_j \to u$ in H. We claim that $v = Au$. To see this, we use point 2):

$$\|Au_j - Au\| \leqslant \alpha \|u_j - u\| \underset{j\to\infty}{\longrightarrow} 0, \tag{4.90}$$

which implies that $v = \lim_{j\to\infty} v_j = Au$.

4. We now prove that $R(A) = H$. Suppose that this is false. Then, since $R(A)$ is closed, one can write the orthogonal decomposition:

$$H = R(A) \oplus R(A)^\perp \tag{4.91}$$

Let $w \in R^\perp(A)$, $w \neq 0$. We have:

$$\beta \|w\|^2 \leqslant B[w,w] = (Aw, w) = 0, \tag{4.92}$$

where the last identity follows from the fact that $Aw \in R(A)$ and $w \in R^\perp(A)$. This implies that $w = 0$, which gives a contradiction. Hence, $H = R(A)$.

5. By Riesz representation theorem, there exists $w \in H$ such that

$$\langle f, v \rangle = (w, v), \qquad \forall v \in H. \tag{4.93}$$

Since the range of A is H, and since A is one-to-one, there exists $u \in H$ such that $Au = w$. Therefore,

$$B[u,v] = (Au, v) = (w, v) = \langle f, v \rangle \,. \tag{4.94}$$

Hence, there exists $u \in H$ such that

$$B[u,v] = \langle f, v \rangle \qquad \forall v \in H. \tag{4.95}$$

6. To conclude, let us prove uniqueness. Let u_1, u_2 be two solutions of Eq. (4.95). Then,

$$B[u_1, v] = B[u_2, v] = \langle f, v \rangle \Rightarrow B[u_1 - u_2, v] = 0 \qquad \forall v \in H. \tag{4.96}$$

Let $v = u - \tilde{u}$. Then

$$\beta \|u_1 - u_2\|^2 \leqslant B[u_1 - u_2, u_1 - u_2] = 0 \,, \tag{4.97}$$

which gives a contradiction.

□

4.2.3 First existence theorem

Sobolev embeddings and Poincaré inequality

In this section, we shall discuss inequalities that will allow us to prove the assumptions of the Lax-Milgram theorem, in the PDE context we are interested in.

The difficult part is the lower bound, $\beta\|u\|^2 \leqslant B[u,u]$. We are looking for an inequality that allows to bound from below L^p norms of Du with L^q norms of u. As we shall see, this cannot be true for all p, q. Let $u \in C_c^\infty(\mathbb{R}^n)$, $u \neq 0$, $u_\lambda(x) := u(\lambda x)$. Suppose that there exists $C > 0$, *independent of* λ, such that:

$$\|u_\lambda\|_{L^q(\mathbb{R}^n)} \leqslant C\|Du_\lambda\|_{L^p(\mathbb{R}^n)} \ . \tag{4.98}$$

By a change of variables,

$$\begin{aligned} \|u_\lambda\|_{L^q(\mathbb{R}^n)} &= \left(\int dx |u_\lambda(x)|^q\right)^{\frac{1}{q}} = \left(\frac{1}{\lambda^n}|u(x)|^q\right)^{\frac{1}{q}} \equiv \left(\frac{1}{\lambda^n}\right)^{\frac{1}{q}} \|u\|_{L^q(\mathbb{R}^n)}, \\ \|Du_\lambda\|_{L^p(\mathbb{R}^n)} &= \left(\int dx |Du_\lambda(x)|^p\right)^{\frac{1}{p}} = \left(\frac{\lambda^p}{\lambda^n}|u(x)|^p\right)^{\frac{1}{p}} \equiv \left(\frac{\lambda}{\lambda^{\frac{n}{p}}}\right) \|Du\|_{L^p(\mathbb{R}^n)} \ . \end{aligned} \tag{4.99}$$

Therefore, Eq. (4.98) implies:

$$\|u\|_{L^q(\mathbb{R}^n)} \leqslant C\lambda^{1-\frac{n}{p}+\frac{n}{q}}\|Du\|_{L^p(\mathbb{R}^n)}. \tag{4.100}$$

Thus, *if* $1 - \frac{n}{p} + \frac{n}{q} \neq 0$, by taking either $l \to 0$ or $l \to \infty$, Eq. (4.100) would imply $\|u\|_{L^p(\mathbb{R}^n)} \leqslant 0$, that is $u = 0$, which is a contradiction.

Therefore, we might only hope to prove Eq. (4.98) for:

$$1 + \frac{n}{q} - \frac{n}{p} = 0 \Rightarrow \frac{1}{q} = \frac{1}{p} - \frac{1}{n}. \tag{4.101}$$

Let $1 \leqslant p < n$. We define the *Sobolev conjugate* of p as the number $q \equiv p^*$ for which Eq. (4.101) holds true:

$$p^* := \frac{np}{n-p} \ . \tag{4.102}$$

Notice that $p^* > p$.

Theorem 4.7 (Gagliardo-Nirenberg-Sobolev inequality)**.** *Let* $1 \leqslant p < n$. *There exists* $C \equiv C(n,p)$ *such that*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leqslant C\|Du\|_{L^p(\mathbb{R}^n)}, \qquad \forall u \in C_c^1(\mathbb{R}^n). \tag{4.103}$$

The proof will be based on the generalized Hölder inequality:

$$\int_U |\prod_{i=1}^m u_i| dx \leqslant \prod_{i=1}^m \|u_i\|_{L^{p_i}(U)}, \qquad \sum_{i=1}^m \frac{1}{p_i} = 1. \tag{4.104}$$

Remark 4.7. *The proof crucially relies on the fact that* u *is compactly supported: the inequality is trivially false if* $u = 1$. *However, the constant* C *does not depend on the support of* u.

Proof. Let us start with the case $p = 1$. Using the compact support of u,

$$u(x) = \int_{-\infty}^{x_i} dy_i u_{x_i}(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n) \tag{4.105}$$

thus

$$|u(x)| \leqslant \int_{-\infty}^{\infty} dy_i |Du(x_1, \cdots, y_i, \cdots, x_n)|. \tag{4.106}$$

For $p=1$ the Sobolev conjugate of p is $p^* = \frac{n}{n-1}$. Therefore, it is natural to consider:

$$|u(x)|^{\frac{n}{n-1}} \leqslant \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} dy_i |Du(x_1, \cdots, y_i, \cdots, x_n)| \right)^{\frac{1}{n-1}} . \tag{4.107}$$

We have:

$$\begin{aligned}
\int_{-\infty}^{\infty} dx_1 |u(x)|^{\frac{n}{n-1}} &\leqslant \int_{-\infty}^{\infty} dx_1 \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} dy_i |Du(x_1, \cdots, y_i, \cdots)| \right)^{\frac{1}{n-1}} = \\
&= \int_{-\infty}^{\infty} dx_1 \left(\int_{-\infty}^{\infty} dy_1 |Du(y_1, \cdots)| \right)^{\frac{1}{n-1}} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} dy_i |Du(x_1, \cdots, y_i, \cdots)| \right)^{\frac{1}{n-1}} \\
&= \left(\int_{-\infty}^{\infty} dy_1 |Du(y_1, \cdots)| \right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} dx_1 \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} dy_i |Du(x_1, \cdots, y_i, \cdots)| \right)^{\frac{1}{n-1}} .
\end{aligned} \tag{4.108}$$

Let us now apply the generalized Hölder inequality, with $p_i = n-1$. We have

$$\int_{-\infty}^{\infty} dx_1 |u(x)|^{\frac{n}{n-1}} \leqslant \left(\int_{-\infty}^{\infty} dy_1 |Du(y_1, \cdots)| \right)^{\frac{1}{n-1}} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} dx_1 dy_i |Du(x_1, \cdots, y_i, \cdots)| \right)^{\frac{1}{n-1}} . \tag{4.109}$$

Next, let us integrate over x_2. We get:

$$\begin{aligned}
&\int_{-\infty}^{\infty} dx_1 dx_2 |u(x)|^{\frac{n}{n-1}} \\
&\leqslant \int dx_2 \left(\int_{-\infty}^{\infty} dy_1 |Du(y_1, \cdots)| \right)^{\frac{1}{n-1}} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} dx_1 dy_i |Du(x_1, \cdots, y_i, \cdots)| \right)^{\frac{1}{n-1}} \\
&= \left(\int_{-\infty}^{\infty} dx_1 dy_2 |Du(x_1, y_2 \cdots)| \right)^{\frac{1}{n-1}} \int dx_2 \left(\int_{-\infty}^{\infty} dy_1 |Du(y_1, x_2, \cdots)| \right)^{\frac{1}{n-1}} \cdot \\
&\quad \cdot \prod_{i=3}^{n} \left(\int_{-\infty}^{\infty} dx_1 dy_i |Du(x_1, x_2, \cdots, y_i, \cdots)| \right)^{\frac{1}{n-1}} .
\end{aligned} \tag{4.110}$$

Using again the generalized Hölder inequality for the x_2 integration, choosing $p_i = n-1$, we have

$$\begin{aligned}
(4.110) \leqslant &\left(\int_{-\infty}^{\infty} dx_1 dx_2 |Du(x_1, x_2 \cdots)| \right)^{\frac{2}{n-1}} \cdot \\
&\cdot \prod_{i=3}^{n} \left(\int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{\infty} dx_1 dy_i |Du(x_1, \cdots, y_i, \cdots, x_n)| \right)^{\frac{1}{n-1}} .
\end{aligned} \tag{4.111}$$

Iterating the same procedure n times (*i.e.* integrating again over $dx_3, \cdots, dx_n$) we finally get

$$\int dx_1 \cdots dx_n |u(x)|^{\frac{n}{n-1}} \leqslant \left(\int dx_1 \cdots dx_n |Du(x_1, \cdots, x_n)| \right)^{\frac{n}{n-1}} , \tag{4.112}$$

which proves the inequality for $p = 1$.

Let us now consider $1 < p < n$. Let $v := |u|^\gamma$, $\gamma > 1$ to be chosen later. By Eq. (4.112), we have

$$\left(\int |u|^{\frac{\gamma n}{n-1}} dx\right)^{\frac{n-1}{n}} \leqslant \int |D|u|^\gamma| dx = \gamma \int |u|^{\gamma-1}|Du| dx \leqslant \gamma \left(\int |u|^{(\gamma-1)\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}} \left(\int |Du|^p\right)^{\frac{1}{p}}, \tag{4.113}$$

where in the last step we used the Hölder with $q = p/(p-1)$. Now let us choose γ such that $\frac{\gamma n}{n-1} = (\gamma - 1)\frac{p}{p-1}$. That is,

$$\gamma\left(\frac{n}{n-1} - \frac{p}{p-1}\right) = -\frac{p}{p-1} \Rightarrow \gamma\left(\frac{p-n}{(n-1)(p-1)}\right) = -\frac{p}{p-1}, \tag{4.114}$$

i.e. $\gamma = p(n-1)/(n-p) > 1$. Plugging this choice into Eq. (4.113) we get:

$$\left(\int |u|^{\frac{\gamma n}{n-1}}\right)^{\frac{n-1}{n}} \left(\int |u|^{\frac{\gamma n}{n-1}}\right)^{-\frac{p-1}{p}} \leqslant \gamma \left(\int |Du|^p\right)^{\frac{1}{p}}, \tag{4.115}$$

with

$$\frac{n-1}{n} - \frac{p-1}{p} = \frac{p(n-1) - n(p-1)}{np} = \frac{n-p}{np} \equiv \frac{1}{p^*}, \tag{4.116}$$

and hence

$$\frac{\gamma n}{n-1} = \frac{pn}{n-p} = p^* . \tag{4.117}$$

We conclude that:

$$\left(\int dx |u|^{p^*}\right)^{\frac{1}{p^*}} \leqslant \gamma \left(\int dx |Du|^p\right)^{\frac{1}{p}}, \qquad 1 < p < n , \tag{4.118}$$

which is what we wanted to prove. □

This inequality can be used to prove that, in some cases, Sobolev spaces are *embedded* in L^q spaces.

Theorem 4.8. *Let $U \subset \mathbb{R}^n$ open and bounded. Let $u \in W_0^{1,p}(U)$, $1 \leqslant p < n$. Then*

$$\|u\|_{L^q(U)} \leqslant C\|Du\|_{L^p(U)}, \qquad \forall q \in [1, p^*], \tag{4.119}$$

with $C \equiv C(p, q, U)$.

Remark 4.8. 1. *In particular, for $q = p$ is allowed, since $p^* > p$. We have:*

$$\|u\|_{L^p(U)} \leqslant C\|Du\|_{L^p(U)}, \tag{4.120}$$

which takes the name of Poincaré inequality.

2. *The Poincaré inequality allows us to prove that on $W_0^{1,p}(U)$, the norms $\|Du\|_{L^p(U)}$ and $\|u\|_{W_0^{1,p}(U)}$ are equivalent. In fact, one trivially has:*

$$\|Du\|_{L^p(U)} \leqslant \|u\|_{W_0^{1,p}(U)} \tag{4.121}$$

and, by Poincaré inequality:

$$\|u\|_{W_0^{1,p}(U)} \leqslant \left(\|u\|^p_{L^p(U)} + \|Du\|^p_{L^p(U)}\right)^{\frac{1}{p}} \leqslant C\|Du\|_{L^p(U)} . \tag{4.122}$$

3. *Theorem 4.8 is telling us that*

$$u \in W_0^{1,p}(U) \Rightarrow u \in L^q(U), \qquad \forall q \in [1, p^*]. \tag{4.123}$$

We stress that the smallest such $L^q(U)$ space is $L^{p^*}(U)$. Indeed, by Hölder:

$$\|u\|_q = \left(\int_U dx|u(x)|^q\right)^{\frac{1}{q}} \leqslant \left(\int_U dx|u(x)|^{qp}\right)^{\frac{1}{qp}} \left(\int_U dx\right)^{\frac{1}{p'}} \leqslant C\|u\|_{p^*}, \tag{4.124}$$

where $\frac{1}{p'} + \frac{1}{p} = 1$ and $p = \frac{p^*}{q} > 1$. We say that the space $W_0^{1,p}(U)$ is embedded in $L^{p^*}(U)$, $p^* = \frac{np}{n-p}$, $1 \leqslant p < n$.

Proof. Let $u \in W_0^{1,p}(U)$. Then, there exists $\{u_m\}_{m\in\mathbb{N}}$, $u_m \in C_c^\infty(U)$ such that $u_m \to u$ in $W^{1,p}(U)$. Let us extend u_m to $\mathbb{R}^n$, setting $u_m = 0$ on $\mathbb{R}^n \backslash U$. By the GNS inequality,

$$\|u_m - u_l\|_{p^*} \leqslant C\|D(u_m - u_l)\|_p \underset{m,l\to\infty}{\longrightarrow} 0\ . \tag{4.125}$$

Thus, $\{u_m\}$ is a Cauchy sequence in $L^{p^*}(U)$, and hence $u_m \to \tilde{u}$ in $L^{p^*}(U)$. Being U bounded, $\tilde{u} \in L^q(U)$, $\forall q : 1 \leqslant q \leqslant p^*$. In particular, $\tilde{u} \in L^p(U)$, which shows that $u = \tilde{u}$, and therefore that $u \in L^q$ for all $q \in [1, p^*]$.

By the GNS inequality:

$$\|u_m\|_{L^{p^*}(U)} \leqslant C\|Du_m\|_{L^p(U)}\ . \tag{4.126}$$

Then, by convergence in $W^{1,p}(U)$:

$$\|Du_m\|_{L^p(U)} = \|D(u_m - u + u)\|_{L^p(U)} \underset{m\to\infty}{\longrightarrow} \|Du\|_{L^p(U)}. \tag{4.127}$$

Also,

$$\|u_m\|_{L^{p^*}(U)} \geqslant C\|u_m\|_{L^q(U)}, \qquad \forall 1 \leqslant q \leqslant p^*, \tag{4.128}$$

and

$$\|u_m\|_{L^q(U)} = \|u_m - u + u\|_{L^q(U)} \underset{m\to\infty}{\longrightarrow} \|u\|_{L^q(U)}, \tag{4.129}$$

by convergence in $L^{p^*}(U)$. All in all:

$$\|u\|_{L^q(U)} \leqslant C\|Du\|_{L^p(U)}, \tag{4.130}$$

for some $C \equiv C(U, n, p)$. □

Proof of existence and uniqueness of the solution

We are now in the position to apply the Lax-Milgram theorem to the weak formulation of the problem (4.62). Let $H \equiv H_0^1(U)$ and

$$B[u, v] = \int_U \Big(\sum_{i,j=1}^n a^{ij}(x) u_{x_i} v_{x_j} + \sum_{j=1}^n b^i(x) u_{x_i} v + c(x)u(x)v \Big) dx, \tag{4.131}$$

where $u, v \in H_0^1(U)$, $a, b, c \in L^\infty(U)$.

Theorem 4.9. *There exist $\alpha, \beta > 0$ and $\gamma \geqslant 0$ such that*

$$|B[u,v]| \leqslant \alpha \|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)}, \tag{4.132}$$

and

$$\beta \|u\|^2_{H_0^1(U)} \leqslant B[u,u] + \gamma \|u\|_{H_0^1(U)}, \tag{4.133}$$

$\forall u, v \in H_0^1(U)$.

Proof. 1. By Cauchy-Schwarz:

$$|B[u,v]| \leqslant C \|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)} \ .$$

2. Recall the uniform ellipticity condition:

$$\sum_{i,j} a^{ij}(x) \xi^i \xi^j \geqslant \theta |\xi|^2, \qquad \forall \xi \in \mathbb{R}^n, \quad \theta > 0. \tag{4.134}$$

Then:

$$\int_U \sum_{i,j=1}^n a^{ij}(x) u_{x_i} u_{x_j} \geqslant \theta \int_U dx |Du|^2. \tag{4.135}$$

and also:

$$\Big| \int_U \sum_{i,j=1}^n a^{ij}(x) u_{x_i} u_{x_j} \Big| \leqslant B[u,u] + \sum_i \|b^i\|_\infty \int |Du||u| dx + \|c\|_\infty \int u^2 dx \ . \tag{4.136}$$

Using that, for all $\epsilon > 0$, $ab \leqslant \epsilon a^2 + \frac{1}{4\epsilon} b^2$, we get:

$$\int_U |Du||u| \leqslant \epsilon \int_U |Du|^2 + \frac{1}{4\epsilon} \int |u|^2. \tag{4.137}$$

Choosing ϵ small enough, Eqs. (4.135), (4.136) imply:

$$\frac{\theta}{2} \int dx |Du|^2 \leqslant B[u,u] + C \int u^2 dx, \tag{4.138}$$

for some $C > 0$. Furthermore, by Poincaré inequality:

$$\frac{\theta}{2} \int dx |Du|^2 \geqslant C' \left(\|u\|^2_{L^2(U)} + \|Du\|^2_{L^2(U)} \right), \tag{4.139}$$

for some $C' > 0$. Thus, Eqs. (4.138), (4.139) imply that there exists $\beta > 0$ such that:

$$\beta \|u\|^2_{H_0^1(U)} \leqslant B[u,u] + \gamma \|u\|_2^2. \tag{4.140}$$

□

Theorem 4.10 (1st Existence Theorem)**.** *There exists $\gamma \geqslant 0$ such that $\forall \mu \geqslant \gamma$ and $\forall f \in L^2(U)$ there exists a unique weak solution $u \in H_0^1(U)$ of*

$$\begin{cases} Lu + \mu u = f, & \text{in } U, \\ u = 0, & \text{on } \partial U. \end{cases} \tag{4.141}$$

Proof. 1. Let γ as in Theorem 4.9, and let $\mu \geqslant \gamma$. We define:

$$B_\mu[u,v] = B[u,v] + \mu(u,v). \tag{4.142}$$

By Theorem 4.9, $B_\mu[u,v]$ satisfies the hypothesis of the LM theorem.

2. Let $f \in L^2(U)$. Then, $\langle f, v\rangle \equiv (f,v)_{L^2(U)}$ defines a bounded linear functional in L^2, hence in $H_0^1(U) \subset L^2(U)$. By LM theorem, there exists a unique $u \in H_0^1(U)$ such that

$$B_\mu[u,v] = \langle f, v\rangle\,, \qquad \forall v \in H_0^1(U). \tag{4.143}$$

This concludes the proof.

□

Remark 4.9. *The value of γ depends on the specific equation. Suppose for instance that $b = 0$, $c \geqslant 0$. Then*

$$B[u,u] \geqslant \sum_{i,j} \int dx a^{ij}(x) u_{x_i} u_{x_j} \geqslant \theta \int |Du|^2 \geqslant \theta C \|u\|_{H_0^1(U)} \tag{4.144}$$

where the second inequality follows from uniform ellipticity and the last from Poincaré inequality. This shows that we can choose $\gamma = 0$, and hence the assumptions of the LM theorem are fulfilled for $B_0[u,v] \equiv B[u,v]$.

Appendix A

Elements of the theory of distributions

In this appendix we shall give a quick introduction to the theory of distributions. We refer the reader to, *e.g.*, [Lieb-Loss] for a more details. A *distribution* is a continuous linear functional on the space of test functions. That is,

$$T : C_c^\infty(U) \to \mathbb{C}, \tag{A.1}$$

such that

$$T(\phi_1 + \phi_2) = T(\phi_1) + T(\phi_2), \qquad T(\lambda\phi) = \lambda T(\phi). \tag{A.2}$$

To define continuity, we need a topology. We say that $\{\phi_n\}_{n\in\mathbb{N}}$, $\phi_n \in C_c^\infty(U)$ converges to $\phi \in C_c^\infty(U)$, $\phi_n \to \phi$, if $\mathrm{supp}(\phi_n - \phi) \subset K \subset U$ where K is compact, and

$$\sup_{x\in K} |D^\alpha \phi_m(x) - D^\alpha \phi(x)| \underset{m\to 0}{\longrightarrow} 0 \; . \tag{A.3}$$

We shall denote by $D(U)$ the space of test functions endowed with the above topology. The space of distributions is the *dual* of $D(U)$, denoted by $D'(U)$. It turns out that $L^p(U)$, and more generally $W^{k,p}(U)$ spaces, can be viewed as spaces of distributions. Given $u \in W^{k,p}(U)$, we define the associated distribution as

$$T(\phi) := \int_U u\phi dx. \tag{A.4}$$

Notice that not all the distributions are of this form. A notable example is the *Dirac delta function*:

$$\delta_x(\phi) := \phi(x) \; . \tag{A.5}$$

It is not difficult to realize that the delta function cannot be written as in Eq. (A.4). Nevertheless, one formally writes:

$$\delta_x(\phi) = \int_U \delta(y-x)\phi(y)dy. \tag{A.6}$$

We say that a sequence of distributions $T^k \in D'(U)$, $k = 1, 2, \ldots$, converges in $D'(U)$ to T, $T^k \to T$, if $T^k(\phi) \to T(\phi)$ for all $\phi \in D(U)$. Also, if T and T' have the form Eq. (A.4), then $T = T'$ if and only if $u = u'$ almost everywhere. We say that $D^\alpha T$ is the distributional derivative of T if:

$$(D^\alpha)T(\phi) = (-1)^{|\alpha|} T(D^\alpha \phi), \qquad \forall \phi \in D(U). \tag{A.7}$$

If T has the form (A.4), this reproduces the definition of weak derivative.

Therefore, Sobolev spaces can also be thought as *spaces of distributions*. If T has the form (A.4) for some u, one often uses the notation $T \equiv u$. One says that u, g are equal in the sense of distributions if

$$\int u\phi = \int g\phi, \qquad \forall \phi \in D(U). \tag{A.8}$$

In particular, $T = \delta_x$ in the sense of distributions means that

$$\int u(y)\phi(y) = \int dy\delta(x-y)\phi(y) = \phi(x), \qquad \forall \phi \in D(U). \tag{A.9}$$

It turns out that the Green function G of the Laplacian $-\Delta$ defines a distribution, which solves the equation:

$$-\Delta G_x = \delta_x \ , \tag{A.10}$$

with $G_x(y) \equiv G(x-y)$.

Appendix B

Elements of the theory of Hilbert spaces

In this appendix we recall some basic notions of the theory of Hilbert spaces.

Definition 39 (Hilbert space)**.** *Let H be a real linear space. We say that H is a Hilbert space if it is a Banach space endowed with an inner product, that generates the norm.*

An inner product $(\cdot,\cdot) : H \times H \to \mathbb{R}$ is a mapping with the following properties:

1. $(u, v) = (v, u)$, $\forall u, v \in H$.
2. $u \to (u, v)$ *is linear.*
3. $(u, u) \geqslant 0$, $\forall u \in H$.
4. $(u, u) = 0 \Leftrightarrow u = 0$.

The norm associated with the inner product is

$$||u||_H = (u, u)^{\frac{1}{2}} \tag{B.1}$$

Example B.0.1. 1. $L^2(U)$ is a Hilbert space, with inner product $(f, g) = \int_U fg$.

2. $H^k(U)$ is a Hilbert space, with inner product $(f, g) = \sum_{|\alpha|\leqslant k} \int_U D^\alpha f D^\alpha g dx$.

Definition 40 (Orthogonal complement)**.** *Let H be a Hilbert space and $K \subset H$ a closed subspace of H. The orthogonal complement of K is defined as*

$$K^\perp := \{u \in H \mid (u, v) = 0 \quad \forall v \in K\}. \tag{B.2}$$

Proposition 11. *The orthogonal complement $K^\perp$ of $K \subset H$ is a closed subspace of H.*

Proof. Let $\{u_k\} \in K^\perp$ such that $\|u_k - u\| \underset{k\to\infty}{\longrightarrow} 0$. Let $v \in K$, then

$$|(u, v)| = |(u, v) - (u_k, v)| \leqslant \|u_k - u\|\|v\| \underset{k\to\infty}{\longrightarrow} 0, \tag{B.3}$$

where we used the orthogonality between u_k and v and the Cauchy-Schwarz inequality. $\square$

Theorem B.1 (Orthogonal decomposition)**.** *Let $K \subset H$ be a closed subspace of H. Then, every $u \in H$ can be uniquely represented as*

$$u = v + w, \tag{B.4}$$

with $v \in K$ and $w \in K^{\perp}$.

Remark B.1.

- *One also writes $H = K \oplus K^{\perp}$.*
- *This decomposition induces the linear mappings:*

$$\begin{aligned} &P_K : H \to K, \qquad P_K u = v, \\ &P_{K^{\perp}} : H \to K^{\perp}, \qquad P_{K^{\perp}} u = w \,. \end{aligned} \tag{B.5}$$

The linear operators $P_k, P_k^{\perp}$ are called orthogonal projections on $K, K^{\perp}$.

Proof. If $u \in K$, then $u = v, w = 0$. If $u \notin K$ we proceed as follows. Let $\{v_k\}_{k\in\mathbb{N}}$, $v_k \in K$ such that

$$\|v - v_k\| \underset{k\to\infty}{\longrightarrow} \inf_{v\in K} \|u - v\|^2. \tag{B.6}$$

We write:

$$\|u - v\|^2 = F(u) + \|u\|^2, \qquad F(v) = \|v\|^2 - 2(u, v). \tag{B.7}$$

Therefore, $F(v_k) \underset{k\to\infty}{\longrightarrow} \inf_{v\in K} F(v) \equiv \alpha$. We would like to prove that $v_k \to v$. To prove this, we write

$$\begin{aligned} F(v_k) + F(v_l) &= \|v_k\|^2 - 2(u, v_k) + \|v_l\| - 2(u, v_l) = \\ &= \frac{1}{2}(\|v_k + v_l\|^2 + \|v_k - v_l\|^2) - 2(u, v_k + v_l) = \\ &= 2\left\|\frac{v_k + v_l}{2}\right\|^2 - 4\left(u, \frac{v_k + v_l}{2}\right) + \frac{1}{2}\|v_k - v_l\|^2 = \\ &\equiv 2F\left(\frac{v_k + v_l}{2}\right) + \frac{1}{2}\|v_k - v_l\| \geqslant 2\alpha + \frac{1}{2}\|v_k - v_l\|. \end{aligned} \tag{B.8}$$

But since $F(v_k)$, $F(v_l) \underset{k,l\to\infty}{\longrightarrow} \alpha$, we have $\|v_k - v_l\| \underset{k,l\to\infty}{\longrightarrow} 0 \Rightarrow \{v_k\}$ is a Cauchy sequence, meaning that $v_k \to v$ in K. Our goal is to prove that v is such that $u - v \in K^{\perp}$. Let $\tilde{v} \in K$. Then

$$f(t) := F(v + t\tilde{v}) \geqslant F(v), \qquad \forall t \in \mathbb{R} \Rightarrow 0 = f'(0) = 2(v - u, \tilde{v}), \quad \forall \tilde{v} \in K, \tag{B.9}$$

where we used

$$\frac{d}{dt} f(t)\Big|_{t=0} = \frac{d}{dt}\|v + t\tilde{v}\| - 2(u, v + t\tilde{v})\Big|_{t=0} = 2(v, \tilde{v}) - 2(u, \tilde{v}) = 2(v - u, \tilde{v}). \tag{B.10}$$

Therefore, $(v - u, \tilde{v}) = 0$ for any $\tilde{v} \in K \Rightarrow v - u \in K^{\perp}$. We are left with proving that the decomposition is unique. Suppose it is not. Then:

$$u = v_1 + w_1 = v_2 + w_2, \qquad v_i \in K, \quad w_i \in K^{\perp}, \quad i = 1, 2. \tag{B.11}$$

Thus, $\forall \tilde{v} \in K$:

$$(u, \tilde{v}) = (v_1, \tilde{v}) = (v_2, \tilde{v}) \Rightarrow 0 = (v_1 - v_2, \tilde{v}) \quad \forall v \in K \Rightarrow v_1 - v_2 \in K^{\perp}. \tag{B.12}$$

But K is a linear space, hence $v_1 - v_2 \in K$. Since $K \cap K^{\perp} = \{0\}$, $v_1 - v_2 = 0$. $\square$

Definition 41 (Orthonormal basis)**.** *A family* $\{w_k\}_{k=1}^{\infty}$, $w_k \in H$, *is called an orthonormal basis of* H *if*

1. $(w_k, w_l) = 0 \quad \forall k \neq l.$
2. $(w_k, w_k) = \|w_k\|^2 = 1$
3. $u = \sum_{i=1}^{\infty}(u, w_i)w_i \quad \forall u \in H$

In particular,

$$\|u\|^2 = \sum_k (u, w_k)^2. \tag{B.13}$$

Definition 42 (Dual space)**.** *We denote by* H^* *the set of all bounded linear functionals on* H.

Remark B.2. *Recall that* $T : H \to \mathbb{R}$ *is a linear functional if* $T(\lambda u + \mu v) = \lambda T(u) + \mu T(v)$. *We say that* T *is bounded if*

$$\|T\| := \sup\{|T(u)| \mid u \in H, \|u\| \leqslant 1\} < \infty\ . \tag{B.14}$$

H^* *is the set of all such maps* T. *One also uses the notation* $\langle T, u\rangle$ *to denote the real number* $T(u)$. $\langle\cdot,\cdot\rangle$ *is called the pairing of* H *and* H^*.

Theorem B.2 (Riesz representation theorem)**.** *For each* $T \in H^*$ *there exists a unique* $u \in H$ *such that*

$$\langle T, v\rangle = (u, v) \qquad \forall v \in H. \tag{B.15}$$

The mapping $T \to u$ *is a linear isomorphism of* H^* *onto* H.