The Symmetric group, its representations and combinatorics: An overview

1. **Introduction:** We will look at a variety of equivalent defitions of the symmetric group (permutation group) on n letters S_n. In particular:

   (a) *Combinatorial definition:* We define S_n as the set of permutations acting on an ordered set of n letters

   (b) *Algebraic definition:* We define S_n as a Coxeter group, that is as the words in an alphabet of generators τ_i subject to some relations between each other.

   (c) *Geometric definition:* We define S_n as the group of reflections of an $n-1$-dimensional simplex about certain symmetry axes (this definition follows naturally from the Coxeter defintion). In showing that this definition of S_n is cpompletely equivalent to the combinatorial one, we will need a result about transpositions, which we will prove using graph theory

   In doing so, we will see that the familiar way of looking at the symmetric group is actually the combinatorial method, giving a strong motivation for wanting to look at the symmetric group from a combinatorial viewpoint.

2. **Representation theory:** A representation of a group G is a map φ from G to the space of invertible endomorphisms on some N-dimensional vector space V, $\varphi : \mathsf{G} \to \mathsf{GL}(V)$, that preserves the group structure. Therefore, studying a representation of a group allows us to study the group using tools from linear algebra, a field of mathematics that is well understood. We will:

   (a) Formally define a representation of a group, a subrepresentation and an *irreducible representation* (the latter will turn out to be the building blocks of all other representations, *c.f. Maschke's Theorem*).

   (b) Define characters of a representation. This allows us to easily present some useful results about representations of finite groups, in particular the symmetric group.

   (c) Using the tools we learned so far, we will define *Young diagrams* and *Young tableaux*, which are purely combinatorial objects that help us classify the irreducible representations of the symmetric group S_n

3. **Particular results in representation theory:** We will now focus on three particular results from representation theory and prove them only using methods from combinatorics. In particular, we will be introduced to combinatorial algorithms for tableaux and bijective proofs:

   (a) For any finite group G, the mutiplicities of the representations of G sum to number of elements in G. This result is usually proven using the theory of characters. However, for the symmetric group S_n, each representation corresponds to a Young diagram $\mathbf{Y}$ consisting of n boxes, and the multiplicity of the representation is given by the number of Young tableaux corresponding to a particular diagram $\mathbf{Y}$, $f^{\mathbf{Y}}$.

Thus,

$$\sum_{\mathbf{Y} \vdash n} (f^{\mathbf{Y}})^2 = |S_n| \ . \tag{1}$$

Since Young diagrams and Young tableaux are combinatorial objects, we will give a bijective proof of eq. (1). A particular algorithm we will require for this is the *Robinson-Schensted correspondence*, which maps permutations to pairs of Young tableaux of the same shape in a bijective manner.

(b) Result 3a utilizes the number of Young tableaux corresponding to a particular Young diagram. This number is given by the *Hook length formula*, which turns out to be an important formula by itself, as it enters into several other quantities used in the representation theory of S_n (and even $\mathsf{SU}(N)$). We will find a bijective proof for the Hook length formula.

(c) Lastly, we find a counting argument for the total number of Young tableaux consisting of n boxes (where $n \in \mathbb{N}$): In the proof of formula (1), we will have seen two ways of constructing a Young tableau from a particular permutation $\rho \in S_n$ (*c.f. the Robinson-Schensted correspondence*). This allows one to establish an equivalence relation, where two permutations are considered to be in the same equivalence class if and only if they produce the same Young tableau. These equivalence classes have interesting properties, which will then be used to fashion a counting argument for the number of equivalence classes in S_n, and thus the number of Young tableaux consisting of n boxes.