Exercises for "Wave Equations of Relativistic Quantum Mechanics"

Sheet 1

Winter Semester 2018/19

Dr. Matthias Lienert

Exercise 1. *Probability conservation for the Schrödinger equation.*

(a) Show that the Schrödinger equation

$$i\partial_t \psi(t, \mathbf{x}) = \left(-\frac{1}{2m}\nabla^2 + V(t, \mathbf{x})\right)\psi(t, \mathbf{x}) \tag{1}$$

implies the continuity equation

$$\partial_t \rho(t, \mathbf{x}) + \operatorname{div} \mathbf{j}(t, \mathbf{x}) = 0 \tag{2}$$

for the probability density $\rho(t, \mathbf{x}) = |\psi|^2(t, \mathbf{x})$ and the probability current $\mathbf{j}(t, \mathbf{x}) = \frac{1}{m}\mathrm{Im}\, \psi^*(t, \mathbf{x})\nabla\psi(t, \mathbf{x})$. (Here, ∇ and div are to be understood with respect to $\mathbf{x}$ only.)

(b) Let ψ be a smooth solution of the Schrödinger equation. Assume in addition that for every t, we have $\max_{\mathbf{x}\in\partial B_R(0)} R^2|\mathbf{j}(t, \mathbf{x})| \to 0$ for $R \to \infty$. Show that (2) implies that the total probability integral

$$P(t) = \int d^3\mathbf{x}\, |\psi|^2(t, \mathbf{x}). \tag{3}$$

is independent of time, i.e., $\frac{d}{dt}P(t) = 0$.

Exercise 2. *Galilean invariance of the Schrödinger equation.*
Consider the Schrödinger equation (1) with $V = 0$.

(a) Show that (1) is invariant under translations $(t, \mathbf{x}) \to (t' = t + \tau, \mathbf{x}' = \mathbf{x} + \mathbf{a})$. To this end, demonstrate that if $\psi'(t, \mathbf{x})$ satisfies (1) in $t, \mathbf{x}$, then $\psi'(t', \mathbf{x}') = \psi(t' - \tau, \mathbf{x}' - \mathbf{a})$ solves (1) in the primed variables.

(b) Next, show that (1) is invariant under rotations $(t, \mathbf{x}) \to (t' = t, \mathbf{x}' = R\mathbf{x})$ where $R \in O(3)$ is a rotation ($R^T = R^{-1}$). To do this, prove that if $\psi(t, \mathbf{x})$ solves (1), then also $\psi'(t', \mathbf{x}') = \psi(t, R^{-1}\mathbf{x}')$ solves (1) in the primed variables. Use that the Laplacian is rotation invariant.

(c) Finally, consider Galilean boosts $(t, \mathbf{x}) \to (t' = t, \mathbf{x}' = \mathbf{x} + \mathbf{u}t)$. These describe the transition from one frame to another moving with relative velocity $-\mathbf{u}$.

(i) First convince yourself that the transformation rule $\psi'(t', \mathbf{x}') = \psi(t, \mathbf{x}' - \mathbf{u}t)$ does not lead to invariance.

(ii) Now find a function $\Phi(t', \mathbf{x}')$ such that the transformation rule $\psi'(t', \mathbf{x}') = \Phi_{\mathbf{u}}(t', \mathbf{x}')\, \psi(t', \mathbf{x}' - \mathbf{u}t')$ leads to invariance.

Hint: Apply the transformation rule with unknown $\Phi_{\mathbf{u}}$ to a plane wave solution $e^{i\mathbf{k}\cdot\mathbf{x} - \omega(\mathbf{k})t}$ of (1) and compare it to a boosted plane wave where $\mathbf{k}$ is replaced by $\mathbf{k} + m\mathbf{u}$. Demanding that you exactly obtain a plane wave with wave vector $\mathbf{k} + m\mathbf{u}$ allows you to read off a candidate for $\Phi_{\mathbf{u}}$. It remains to show that this indeed leads to invariance.