Exercises for "Wave Equations of Relativistic Quantum Mechanics"

Sheet 2

Winter Semester 2018/19

Dr. Matthias Lienert

Exercise 1. *Transformation behavior of derivatives of scalar fields*

(a) Let $f : \mathbb{M} \to \mathbb{C}$ be a twice differentiable scalar field which transforms under Lorentz transformations $x \to x' = \Lambda x$ as

$$f'(x') = f(x). \tag{1}$$

How do then (i) $\partial_\mu f(x)$ and (ii) $\partial_\mu \partial^\mu f(x)$ transform under Λ?

Exercise 2. *Lorentz group*

(a) Let $\mathbf{v} \in \mathbb{R}^3$, $v = |\mathbf{v}| < 1$ and $\gamma(v) = 1/\sqrt{1-v^2}$. Show that Lorentz boosts

$$\Lambda = \begin{pmatrix} \gamma(v) & \gamma(v)\mathbf{v}^T \\ \gamma(v)\mathbf{v} & \mathbb{1}_3 + \frac{\gamma(v)-1}{v^2}\mathbf{v}\mathbf{v}^T \end{pmatrix}. \tag{2}$$

indeed preserve spacetime distances $s^2(x,y) = (x-y)_\mu (x-y)^\mu$ for all $x, y \in \mathbb{M}$.

(b) First show that the definition of Lorentz transformations Λ (see lecture) is equivalent to

$$\Lambda^\mu{}_\rho \, \eta_{\mu\nu} \, \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}. \tag{3}$$

Then use this definition to prove that the set $\mathcal{L}$ of Lorentz transformations forms a group.

(c) Using (3) show that $\det \Lambda \in \{-1, +1\}$. (*Hint:* First rewrite (3) as $\Lambda^T \, \eta \, \Lambda = \eta$.) Furthermore, demonstrate that $|\Lambda^0{}_0| \geq 1$.

(d) Show that $\mathcal{L}$ is a six-dimensional Lie group.
Hint: Show that every (proper) Lorentz transformation $\Lambda \in \mathcal{L}_+^\uparrow$ can be written as the product of a boost $\Lambda(\mathbf{v})$ with velocity $\mathbf{v} \in \mathbb{R}^3$, $|\mathbf{v}| < 1$, and a rotation $\Lambda(\varphi)$ with rotation vector $\boldsymbol{\varphi} \in \mathbb{R}^3$, $|\boldsymbol{\varphi}| < 2\pi$ as follows:

$$\Lambda = \Lambda(\mathbf{v}) \, \Lambda(\boldsymbol{\varphi}). \tag{4}$$

Then use $\mathbf{v}$ and $\boldsymbol{\varphi}$ as the coordinates of the group manifold for $\Lambda \in \mathcal{L}_+^\uparrow$. The other connected components of $\mathcal{L}$ (see lecture) can be treated similarly.

Exercise 3. *Relativistic formulation of conservation laws*
Let $j : \mathbb{M} \to \mathbb{M}$ be a differentiable vector field which satisfies the continuity equation

$$\partial_\mu j^\mu(x) = 0. \tag{5}$$

Assume furthermore that there is an $R > 0$ such that $j(t, \mathbf{x}) = 0$ for $|\mathbf{x}| > R$. Show that for each pair of space-like hypersurfaces $\Sigma, \Sigma' \subset \mathbb{M}$ we then have:

$$\int_\Sigma d\sigma(x)\, n_\mu(x)\, j^\mu(x) = \int_{\Sigma'} d\sigma(x)\, n'_\mu(x)\, j^\mu(x), \tag{6}$$

where $n(x)$ ($n'(x)$) is the future-pointing unit covector field at Σ (Σ').
Hint: Apply the 4-dimensional divergence theorem to a suitably chosen closed surface $S \subset \mathbb{M}$ which contains $\Sigma \cap \{(x^0, \mathbf{x}) \in \mathbb{M} : |\mathbf{x}| < R\}$ and $\Sigma' \cap \{(x^0, \mathbf{x}) \in \mathbb{M} : |\mathbf{x}| < R\}$.