Exercises for "Wave Equations of Relativistic Quantum Mechanics"

Sheet 3

Winter Semester 2018/19

Dr. Matthias Lienert

Exercise 1. *Current of the Salpeter equation*

(a) Prove that for solution $\psi \in C^1(\mathbb{R}^4)$ with $\psi(t,\cdot), \partial_t\psi(t,\cdot) \in L^2(\mathbb{R}^3)$ of the Salpeter equation
$$i\partial_t\psi(t,\mathbf{x}) = \sqrt{-\Delta + m^2}\,\psi(t,\mathbf{x}) \tag{1}$$
the integral $P(t) = \int d^3\mathbf{x}\,|\psi|^2(t,\mathbf{x})$ is conserved (independent of time).
Hint: Relate $\psi(t,\mathbf{x})$ to its Fourier transform $\widetilde{\psi}(t,\mathbf{k})$ and $\int d^3\mathbf{x}\,|\psi|^2$ to $\int d^3\mathbf{k}\,|\widetilde{\psi}|(t,\mathbf{k})$.

(b) Determine a spatial current $\mathbf{j}(t,\mathbf{x})$ such that $\rho = |\psi|^2$ and $\mathbf{j}$ satisfy the continuity equation. What can you say about the transformation behavior of the combined object $j = (\rho, \mathbf{j})$ under Lorentz transformations?

Exercise 2. *Current of the Klein-Gordon equation*

(a) Prove that the KG equation implies that the 4-current
$$j^\mu(x) = \mathrm{Im}\,(\psi(x)\partial^\mu\psi^*(x)) \tag{2}$$
satisfies the continuity equation $\partial_\mu j^\mu(x) = 0$.

(b) For $\widetilde{\phi}_+(\mathbf{k}) \in L^2(\mathbb{R}^3)$, consider a positive energy solution of the KG equation ($\omega(\mathbf{k}) = \sqrt{\mathbf{k}^2 + m^2}$)
$$\psi(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{x} - i\omega(\mathbf{k})x^0}\,\widetilde{\phi}_+(\mathbf{k}). \tag{3}$$
Let $\Sigma \subset \mathbb{M}$ be a space-like hyperplane.

(i) Beginning with the $t = 0$ hyperplane $\Sigma_{t=0}$ show that
$$P(\Sigma) = \int_\Sigma d\sigma_\mu(x)\,\mathrm{Im}(\psi(x)\partial^\mu\psi^*(x)) \tag{4}$$
is a positive number. (You may use the Fourier representation of the δ-function $\delta^{(3)}(\mathbf{k}) = \frac{1}{(2\pi)^3}\int d^3\mathbf{x}\, e^{i\mathbf{k}\cdot\mathbf{x}}$ here.)

(ii) Use the result of Exercise 3, Sheet 2, to conclude that $P(\Sigma)$ is equal to the same positive number for all space-like hyperplanes $\Sigma \subset \mathbb{M}$ (given the requirements in that exercise).

(c) Show that the KG current (2) can be space-like even for positive energy solutions. Consider the following example (due to Roderich Tumulka, J. Phys. A: Math. Gen. 35 (2002) 7961-7962, `https://arxiv.org/abs/quant-ph/0202140`):

$$\psi = c_1\psi_1 + c_2\psi_2 + c_2\psi_3 \tag{5}$$

where each ψ_j is a plane wave solution with positive energy and wave vector, i.e.

$$\psi_j(x) = e^{-ik_\mu^{(j)}x^\mu} \tag{6}$$

with $(k_\mu^{(1)}) = (m, 0, 0, 0)$, $(k_\mu^{(2)}) = (\sqrt{27}m, \sqrt{26}m, 0, 0)$, $(k_\mu^{(3)}) = (\sqrt{27}m, 0, \sqrt{26}m, 0)$ and $c_1 = 3$, $c_2 = -1/\sqrt{3} - i$, $c_3 = i$.