Exercises for "Wave Equations of Relativistic Quantum Mechanics"

Sheet 4

Winter Semester 2018/19

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Exercise 1. *Green's function of the Klein-Gordon equation in $d = 1$.*

(a) Show first show that

$$\frac{1}{2}\theta(t' - t - |x' - x|) \tag{1}$$

is a Green's function of the wave equation, i.e.

$$(\partial_t^2 - \partial_x^2)\frac{1}{2}\theta(t' - t - |x' - x|) = \delta(t' - t)\delta(x' - x). \tag{2}$$

Hints. You can use $\theta'(x) = \delta(x)$, $\partial_x|x| = \operatorname{sgn}(x)$, $\partial_x \operatorname{sgn}(x) = \partial_x(\theta(x) - \theta(-x)) = \delta(x) - (-\delta(-x)) = 2\delta(x)$, and $\delta(x-a)f(x) = f(a)$. Your calculation will also contain the derivative of the delta function, $\delta'(x)$ (which is well-defined in a distributional sense, namely by $\int dx\, \delta'(x)f(x) = -f'(0)$ for every test function f).

(b) With the help of (a) show that

$$K(t, x; t', x') = \frac{1}{2}\theta(t' - t - |x' - x|)J_0(m\sqrt{(t' - t)^2 - |x' - x|^2}) \tag{3}$$

is a Green's function of the 1+1 dimensional KG equation, i.e.

$$(\partial_t^2 - \partial_x^2)K(t, x; t', x') = \delta(t' - t)\delta(x' - x). \tag{4}$$

Here, J_0 is a Bessel function, defined by

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{x}{2}\right)^{2k}. \tag{5}$$

Hint. You can use $J_0'(x) = -J_1(x)$ and $J_1'(x) = J_0(x) - \frac{1}{x}J_1(x)$ where $J_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+1)!} \left(\frac{x}{2}\right)^{2k+1}$.

(c) Demonstrate that for $T > 0$, K satisfies the conditions

$$K(T, x; t', x') = 0, \quad (\partial_t K)(T, x; t', x') = 0, \quad t' \in (0, T),\ x, x' \in \mathbb{R}. \tag{6}$$

Exercise 2. *Solution formula of the Klein-Gordon equation in $d = 1$.*

(a) Use the Green's function (3) and the general solution formula of the lecture to determine the solution formula of the Cauchy problem

$$\begin{cases} \psi(0,x) = f(x), & x \in \mathbb{R}, \\ \partial_t \psi(0,x) = g(x), & x \in \mathbb{R}, \\ (\partial_t^2 - \partial_x^2 + m^2)\psi(t,x) = 0, & x \in \mathbb{R},\ t \in (0,T). \end{cases} \tag{7}$$

(b) Show that for $m = 0$ one obtains the familiar solution formula of the wave equation,

$$u(\tau, y) = \frac{1}{2}(f(y+\tau) + f(y-\tau)) + \frac{1}{2}\int_{-\tau}^{\tau} dx\ g(x+y). \tag{8}$$

(c) For the special case $f = 0$ prove that the formula of (a) really yields a C^2-solution of (7) if $g \in C^1(\mathbb{R})$. (You can use that J_0 is smooth here.)