Exercises for "Wave Equations of Relativistic Quantum Mechanics"

Sheet 5

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Exercise 1. *Finite propagation speed.*
Prove in detail that the KG equation has finite propagation speed.

Exercise 2. *Energy integral.*
Let $f \in C^3(\mathbb{R}^3)$ and $g \in C^2(\mathbb{R}^3)$ with $f, g, \partial_i f, \partial_i g \in L^2(\mathbb{R}^3)$ $\forall i = 1,2,3$. Using the method of energy integrals (see lecture), prove that the solutions of the Cauchy problem

$$\begin{cases} \psi(0,\mathbf{x}) = f(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^3, \\ \partial_t \psi(0,\mathbf{x}) = g(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^3, \\ \partial_t^2 \psi(t,\mathbf{x}) - \nabla \cdot \frac{1}{1+\mathbf{x}^2} \nabla \psi(t,\mathbf{x}) + e^{-\mathbf{x}^2} \psi(t,\mathbf{x}) = 0, & \mathbf{x} \in \mathbb{R}^3,\ t \in (0,T). \end{cases} \tag{1}$$

are unique. (You do not need to find any solutions.)

Exercise 3. *Cauchy problem on space-like hyperplanes.*
Let $\Sigma \subset \mathbb{M}$ be a space-like hyperplane and let n be the future-pointing unit normal vector field at Σ. Use a Lorentz transformation and the solution formula of the lecture to solve the Cauchy problem

$$\begin{cases} \psi(x) = f(x), & x \in \Sigma, \\ n^\mu \partial_\mu \psi(x) = g(x), & x \in \Sigma, \\ (\Box + m^2)\psi(x) = 0, & x \in \mathbb{M}. \end{cases} \tag{2}$$

Exercise 4. *Energy-momentum tensor of the KG equation*

(a) Show that for every solution $\psi \in C^2(\mathbb{R}^4)$ of the KG equation, the energy-momentum tensor

$$T^{\mu\nu}(x) =$$
$$\frac{1}{2}\left\{(\partial^\nu \psi(x))^*(\partial^\mu \psi(x)) + (\partial^\nu \psi(x))^*(\partial^\mu \psi(x)) - \eta^{\mu\nu}((\partial_\rho \psi^*(x))(\partial^\rho \psi(x)) - m^2|\psi(x)|^2)\right\}. \tag{3}$$

is conserved, meaning

$$\partial_\mu T^{\mu\nu} = 0 \ \forall \nu = 0,1,2,3. \tag{4}$$

(b) Show that the "energy" of the KG equation (see lecture) can be expressed as the spatial integral of T^{00}.