Exercises for "Wave Equations of Relativistic Quantum Mechanics"

Sheet 6

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Exercise 1. *Dirac γ-matrices*

(a) Find a representation of the Clifford algebra relations

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}\, \mathbb{1}, \quad \mu, \nu = 0, ..., d \tag{1}$$

for (i) $d = 1$ and (ii) $d = 2$ using (both times) complex 2×2 matrices.

(b) Check that for $d = 3$ the set of γ-matrices

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \quad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}, \; j = 1, 2, 3. \tag{2}$$

satisfies the relations (1). σ^j, $j = 1, 2, 3$ are the Pauli matrices (see lecture).

(c) For $d = 3$, find a different set of γ-matrices from (2) satisfying the relations (1).

Exercise 2. *Energy-momentum tensor of the Dirac equation.*
Let $\psi \in C^1(\mathbb{R}^4)$ be a solution of the Dirac equation. Show that then

$$T^{\mu\nu}(x) = \frac{i}{4}\overline{\psi}(x) \left[\gamma^\mu \partial^\nu + \gamma^\nu \partial^\mu\right] \psi(x) - \frac{i}{4}\left[(\partial^\nu \overline{\psi}(x))\gamma^\mu \psi(x) + (\partial^\mu \overline{\psi}(x))\gamma^\nu \psi(x)\right] \tag{3}$$

is real-valued, symmetric, i.e. $T^{\mu\nu} = T^{\nu\mu}$, and satisfies the continuity equations

$$\partial_\mu T^{\mu\nu}(x) = 0, \quad \nu = 0, 1, 2, 3. \tag{4}$$

Exercise 3. *Total probability conservation implies the uniqueness of solutions of the Dirac equation*

(a) Let $\psi \in C^1(\mathbb{R}^4, \mathbb{C}^4)$ be a solution of the Dirac equation with $\psi_{|\Sigma} \in L^2(\Sigma, \mathbb{C}^4)$. Show that the integral

$$P(\Sigma) = \int_\Sigma d\sigma(x)\; n_\mu(x)\, \overline{\psi}(x)\gamma^\mu \psi(x) \tag{5}$$

is non-negative and independent of the choice of the space-like hyperplane Σ.

(b) Let $\psi_1, \psi_2 \in C^1(\mathbb{R}^4, \mathbb{C}^4)$ be two solutions of the Dirac equation with $\psi_1(t, \cdot), \psi_2(t, \cdot) \in L^2(\mathbb{R}^3, \mathbb{C}^4)\; \forall t \in \mathbb{R}$. Assume furthermore that both ψ_1, ψ_2 satisfy the initial condition $\psi_i(0, \mathbf{x}) = \psi_0(\mathbf{x})$ for some $\psi_0 \in C^1(\mathbb{R}^4, \mathbb{C}^4) \cap L^2(\mathbb{R}^3, \mathbb{C}^4)$.

Show that $P(\Sigma) = P(\Sigma')$ for all space-like hyperplanes Σ, Σ' implies that $\psi_1(t, \cdot) = \psi_2(t, \cdot)\; \forall t \in \mathbb{R}$ and hence $\psi_1 = \psi_2$.

Hint: Consider equal-time hyperplanes Σ_t in a particular frame first.